-: POWER SERIES :-

• **Introduction** :- A series of the form $a_0+a_1x+a_2x^2+\cdots$ where $a_0, a_1, a_2, \cdots$ are real numbers, is called Power series. The general form of a Power series is $a_0+a_1(x-x_0)+a_2(x-x_0)^2+\cdots$ where $a_0, a_1, a_2, \cdots, x_0 \in R$.

This is called a Power series about the Point x_0.

The general form reduces to the form $a_0+a_1x+a_2x^2+\cdots$ (which is a Power series about 0) by the substitution $x'=x-x_0$.

To study the nature and Properties of a Power series we shall Consider the Power series about 0, i.e., a series of the form

$$a_0+a_1x+a_2x^2+\cdots,$$

This is denoted by $\sum_{n=0}^{\infty} a_nx^n$. It is a series of functions $\sum_{n=0}^{\infty} f_n(x)$ where, for $n=0,1,2,\cdots$, $f_n(x)=a_nx^n$, $x\in R$.

Although each f_n is defined for all real x, it is not expected that the series $\sum_{n=0}^{\infty} a_nx^n$ will converge for all real x.

For example, the series $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ Converge for all real x; while the series $1+x+x^2+\cdots$ Converge only for all $x\in(-1,1)$; and the series $1+x+2!\,x^2+3!\,x^3+\cdots$ Converges only for $x=0$.

It appears that some Power series Converge for all $x\in R$. They are called everywhere Convergent Power series. Some Power series Converge only for $x=0$. They are called nowhere convergent Power series. Some Power series Converge for some real x and diverge for the others.

Note:- We shall use the symbol $\sum_{n=0}^{\infty} a_nx^n$ to denote the Power series $a_0+a_1x+a_2x^2+\cdots$ and also to denote the sum of the series, when it exists.

Theorem:- If a Power series $a_0+a_1x+a_2x^2+\cdots$ Converges for $x=x_1$, then the series Converges absolutely for all real x satisfying $|x|<|x_1|$.

Proof:- Since the Power series $\sum_{n=0}^{\infty} a_nx^n$ Converges for $x=x_1$, $\sum_{n=0}^{\infty} a_nx_1^n$ is Convergent. It follows that $\lim_{n\to\infty} a_nx_1^n=0$. Again the Convergence of the sequence $\{a_nx_1^n\}$ implies that the sequence $\{a_nx_1^n\}$ is bounded.

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Therefore there exists a Positive real number K such that $|a_n x_1^n| \leq K$ for all $n \in N$.

$$|a_n x^n| = |a_n x_1^n| \cdot \left|\frac{x}{x_1}\right|^n \leq K \cdot \left|\frac{x}{x_1}\right|^n.$$

For all real x satisfying $|x| < |x_1|$, $\sum_{n=0}^{\infty} \left|\frac{x}{x_1}\right|^n$ is a Convergent series of Positive real numbers.

By Comparison test, $\sum_{n=0}^{\infty} |a_n x^n|$ is Convergent if $|x| < |x_1|$.

Therefore $\sum_{n=0}^{\infty} a_n x^n$ is absolutely Convergent if $|x| < |x_1|$.

Theorem :- If a Power series $a_0 + a_1 x + a_2 x^2 + \cdots$ diverges for $x = x_1$, then the series diverges for all real x satisfying $|x| > |x_1|$.

Proof :- Let the Power series be Convergent for $x = c$ such that $|c| > |x_1|$. Since the series Converges for $x = c$, the series $\sum_{n=0}^{\infty} a_n c^n$ is Convergent. It follows that $\lim_{n \to \infty} a_n c^n = 0$. Again the Convergence of the sequence $\{a_n c^n\}$ implies that the sequence $\{a_n c^n\}$ is bounded. Therefore there exists a Positive real number K such that $|a_n c^n| \leq K$ for all $n \in N$.

$$|a_n x_1^n| = |a_n c^n| \cdot \left|\frac{x_1}{c}\right|^n \leq K \left|\frac{x_1}{c}\right|^n.$$

For all real x satisfying $|x_1| < |c|$, $\sum_{n=0}^{\infty} \left|\frac{x_1}{c}\right|^n$ is a Convergent series of Positive real numbers.

By Comparison test, $\sum_{n=0}^{\infty} |a_n x_1^n|$ is Convergent if $|x_1| < |c|$.

Therefore $\sum_{n=0}^{\infty} a_n x_1^n$ is Convergent if $|x_1| < |c|$, a Contradiction to the hypothesis.

This Proves that the series is divergent for all real x satisfying $|x| > |x_1|$.

Theorem:- If a Power series $\sum_{n=0}^{\infty} a_n x^n$ be neither nowhere Convergent nor everywhere Convergent, then there exists a Positive real number R such that the series Converges absolutely for all real x satisfying $|x| < R$ and diverges for all real x satisfying $|x| > R$. V.11-02,

Definition :- If a Power series $\sum_{n=0}^{\infty} a_n x^n$ be neither nowhere Convergent nor everywhere Convergent, then there exists a Positive real number R such that the series Converges absolutely for all real x satisfying $|x|<R$ and diverges for all real x satisfying $|x|>R$. R is called the radius of Convergence of the Power Series $\sum_{n=0}^{\infty} a_n x^n$.

note 1 :- We define $R=0$ for a Power series which is nowhere Convergent and $R=\infty$ for a Power series which is everywhere Convergent.

note 2 :- The Convergence of the Power series at $x=R$ and $x=-R$ depends on the nature of the sequence $\{a_n\}$. There are Power series for which both R and $-R$ are Points of Convergence, or both R and $-R$ are Points of divergence, or one of R and $-R$ is a Point of Convergence and the other is a Point of divergence.

Definition :- The interval of Convergence of a Power series in x is the interval of values such that the series Converges for every value of x in this interval and does not Converge for all the others.

Determination of the radius of Convergence :-

Theorem (Cauchy-Hadamard) :-

Let $\sum_{n=0}^{\infty} a_n x^n$ be a Power series and let $\overline{\lim} |a_n|^{1/n} = \mu$. Then

(i) if $\mu=0$, the series is everywhere Convergent.

(ii) if $0<\mu<\infty$, the series is absolutely Convergent for all x satisfying $|x|<\frac{1}{\mu}$ and is divergent for all x satisfying $|x|>1/\mu$.

(iii) if $\mu=\infty$, the series is nowhere Convergent.

Proof :- (i) Let $x_0 \neq 0$ and $\epsilon = \frac{1}{2|x_0|}$.

Since $\overline{\lim} |a_n|^{1/n} = 0$, there exists a natural number K such that $|a_n|^{1/n} < \epsilon$ for all $n \geq K$. or, $|a_n x_0^n| < \frac{1}{2^n}$ for all $n \geq K$.

Since $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is a Convergent series of Positive real numbers, $\sum_{n=0}^{\infty} |a_n x_0^n|$ is a Convergent Series, by Comparison test.

It follows that $\sum_{n=0}^{\infty} a_n x_0^n$ is absolutely Convergent and is therefore Convergent.

As x_0 is arbitrary, the series $\sum_{n=0}^{\infty} a_n x^n$ is everywhere Convergent.

(ii) $\overline{\lim} |a_n x^n|^{1/n} = \overline{\lim} \left(|a_n|^{1/n} |x| \right) = \mu \cdot |x|$.

By Cauchy's root test, the series $\sum_{n=0}^{\infty} |a_n x^n|$ is Convergent if $\mu|x|<1$.

Therefore if $|x| < 1/\mu$, the series $\sum_{n=0}^{\infty} |a_n x^n|$ is Convergent i,e., the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely Convergent.

If $\mu|x|>1$, $\overline{\lim} |a_n x^n|^{1/n} = \overline{\lim} \left(|a_n|^{1/n} |x| \right) = \mu |x| > 1$.

Let $u_n = a_n x^n$. Then $\overline{\lim} |u_n|^{1/n} > 1$ and this implies $\lim_{n\to\infty} |u_n| \neq 0$.

Consequently, $\lim_{n\to\infty} u_n \neq 0$ and it follows that $\sum_{n=0}^{\infty} a_n x^n$ is divergent.

(iii) If possible, let the series $\sum_{n=0}^{\infty} a_n x^n$ be Convergent for $x = x_0 \, (x_0 \neq 0)$.

Then $\lim_{n\to\infty} a_n x_0^n = 0$.

The sequence $\{a_n x_0^n\}$ being a bounded sequence, there exists a Positive real number B such that $|a_n x_0^n| < B$ for all $n \in N$.

This showes that the sequence $\{|a_n|^{1/n}\}$ is a bounded sequence and this Contradicts that $\overline{\lim} |a_n|^{1/n} = \infty$.

Thus the series $\sum_{n=0}^{\infty} a_n x^n$ is not Convergent for $x = x_0$. As x_0 is an arbitrary non-zero real number, the series $\sum_{n=0}^{\infty} a_n x^n$ is nowhere Convergent.

Note:- The radius of Convergence of the series is $\dfrac{1}{\overline{\lim} |a_n|^{1/n}}$.
when $0<\mu<\infty$, $R = 1/\mu$; when $\mu = 0$, $R=\infty$; when $\mu=\infty$, $R=0$.

Theorem:- (Ratio test):-

Let $\sum_{n=0}^{\infty} a_n x^n$ be a Power series and let $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \mu$. Then

(i) if $\mu = 0$, the series is everywhere Convergent.

(ii) if $0<\mu<\infty$ the series is absolutely Convergent for all x satisfying $|x| < \frac{1}{\mu}$ and the series is divergent for all x satisfying $|x| > 1/\mu$.

(iii) if $\mu = \infty$, the series is nowhere Convergent.

Proof:- (i) Let $x \neq 0$ and let $u_n = a_n x^n$.

Then $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x| = \mu \cdot |x| = 0 < 1$.

By D'Alembert's ratio test, the series $\sum |u_n|$ is Convergent.

Therefore $\sum_{n=0}^{\infty} a_n x^n$ is absolutely Convergent for all non-zero real x.

Consequently, $\sum_{n=0}^{\infty} a_n x^n$ is Convergent for all non-zero real x, i.e., the Series $\sum_{n=0}^{\infty} a_n x^n$ is everywhere Convergent.

(ii) Let $x \neq 0$. Then $\lim_{n\to\infty} \left|\frac{a_{n+1} x^{n+1}}{a_n x^n}\right| = \mu |x|$.

By D'Alembert's ratio test, the series $\sum_{n=0}^{\infty} |a_n x^n|$ is Convergent if $|x| < 1/\mu$.

The series is Convergent for $x = 0$ also.

Hence the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely Convergent for all real x satisfying $|x| < 1/\mu$.

When $|x| > 1/\mu$, $\lim_{n\to\infty} \left|\frac{a_{n+1} x^{n+1}}{a_n x^n}\right| > 1$. Let $u_n = a_n x^n$.

Then $\lim \left|\frac{u_{n+1}}{u_n}\right| > 1$. Let $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = l$. Then $l > 1$.

Let us choose $\epsilon > 0$ such that $l - \epsilon > 1$. There exists a natural number m such that $l - \epsilon < \left|\frac{u_{n+1}}{u_n}\right| < l + \epsilon$ for all $n \geq m$.

Therefore $\left|\frac{u_{n+1}}{u_n}\right| > 1$ for all $n \geq m$.

or, $|u_{n+1}| > |u_n|$ for all $n \geq m$.

This shows that the sequence $\{|u_n|\}$ is ultimately a monotone increasing sequence of Positive real numbers and therefore $\lim_{n\to\infty} |u_n|$ Can not be 0.

It follows that $\lim_{n\to\infty} u_n \neq 0$ and Consequently $\sum_{n=0}^{\infty} a_n x^n$ is divergent for all real x satisfying $|x| > \frac{1}{\mu}$.

(iii) Let $u_n = a_n x^n$.

For $x \neq 0$, $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x| = \infty$.

Let us choose $G > 1$. There exists a natural number m such that

$\left|\frac{u_{n+1}}{u_n}\right| > G$ for all $n \geq m$.

Therefore $|u_{n+1}| > |u_n|$ for all $n \geq m$.

This shows that the sequence $\{|u_n|\}$ is ultimately monotone increasing sequence of positive real numbers and therefore $\lim_{n\to\infty} |u_n|$ can not be 0.

It follows that $\lim_{n\to\infty} u_n \neq 0$ and consequently $\sum_{n=0}^{\infty} a_n x^n$ is divergent for all real $x(\neq 0)$.

Thus the series $\sum_{n=0}^{\infty} a_n x^n$ is nowhere convergent.

Note 1:- The radius of convergence of the power series is $\dfrac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$.

Note 2:- We have $\underline{\lim}\left|\frac{a_{n+1}}{a_n}\right| \leq \underline{\lim}|a_n|^{1/n} \leq \overline{\lim}|a_n|^{1/n} \leq \overline{\lim}\left|\frac{a_{n+1}}{a_n}\right|$.

Therefore if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists then $\lim_{n\to\infty}|a_n|^{1/n}$ exists, but the converse is not true.

Q1. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ where

(i) $a_n = \dfrac{(-1)^n \cdot n^n}{n! \cdot 2^n}$, $n = 1, 2, \ldots$, $a_0 = 0$; (ii) $a_n = \dfrac{2^n}{n^2}$, $n = 1, 2, 3, \ldots$, $a_0 = 0$;

(iii) $a_n = \left(\frac{1}{3}\right)^n$ if n be odd.
$\quad = \left(\frac{1}{2}\right)^n$ if n be even; (iv) $a_n = 2^n + 3^n$, $n \geq 1$;

V.H-05,99 (v) $a_0 = 1$, $a_n = (\sqrt[n]{n} + 1)^n$, $n \geq 1$; (vi) $a_0 = 1$, $2 \leq |a_n| \leq 3$ for $n \geq 1$.

Soln:- (i) $\left|\frac{a_{n+1}}{a_n}\right| = \dfrac{(n+1)^{n+1}}{(n+1)! \cdot 2^{n+1}} \cdot \dfrac{n!\, 2^n}{n^n} = \left(1 + \frac{1}{n}\right)^n \cdot \frac{1}{2}$, $n \geq 1$.

Therefore, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{e}{2}$.

This implies that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is $\frac{2}{e}$.

(ii) $\left|\frac{a_{n+1}}{a_n}\right| = \dfrac{2^{n+1}}{(n+1)^2} \cdot \dfrac{n^2}{2^n} = \dfrac{2}{\left(1 + \frac{1}{n}\right)^2}$, $n \geq 1$.

Therefore, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2$.

This implies that the radius of radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is $\frac{1}{2}$.

(iii) Since $a_{2n} = \left(\frac{1}{2}\right)^{2n}$ and $a_{2n-1} = \left(\frac{1}{3}\right)^{2n-1}$ for $n = 1, 2, 3, \ldots$

The sequence $\{|a_n|^{\frac{1}{n}}\}$ is $\{\frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \ldots\}$.

Let $\bar{A}_n = \sup\{|a_n|^{1/n}, |a_{n+1}|^{\frac{1}{n+1}}, |a_{n+2}|^{\frac{1}{n+2}}, \ldots\}$ for $n = 1, 2, 3, \ldots$

Then $\bar{A}_n = \frac{1}{2}$ for all $n = 1, 2, 3, \ldots$

Therefore $\overline{\lim} |a_n|^{1/n} = \frac{1}{2}$.

Thus the radius of Convergence of the Power series $\sum_{n=0}^{\infty} a_n x^n$ is 2.

(iv) Since $a_n = 2^n + 3^n$, $n = 1, 2, 3, \ldots$

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} 3\left\{\left(\tfrac{2}{3}\right)^n + 1\right\}^{1/n} = 3. \quad \left[\text{As } \lim_{n\to\infty}\left(\tfrac{2}{3}\right)^n = 0\right]$$

Thus the radius of Convergence of the Power series $\sum_{n=0}^{\infty} a_n x^n$ is $\frac{1}{3}$.

(v) $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} (n^{1/n} + 1) = 2$. $\left[\text{As } \lim_{n\to\infty} n^{\frac{1}{n}} = 1\right]$

Thus the radius of Convergence of the Power series $\sum_{n=0}^{\infty} a_n x^n$ is $\frac{1}{2}$.

(vi) Since $2 \le |a_n| \le 3$ for all $n \geq 1$. $2^{1/n} \le |a_n|^{1/n} \le 3^{1/n}$.

Therefore, $\lim_{n\to\infty} |a_n|^{1/n} = 1$. $\left[\text{Since } \lim_{n\to\infty} 2^{1/n} = \lim_{n\to\infty} 3^{1/n} = 1\right]$.

Thus the radius of Convergence of the Power series $\sum_{n=0}^{\infty} a_n x^n$ is 1.

Q2. Show that the radius of Convergence of the Power series $\sum_{n=0}^{\infty} a_n x^n$ where $a_{2n} = \frac{1}{3^n}$ and $a_{2n-1} = \frac{1}{3^{n+1}}$ is $\sqrt{3}$. [V.H-00,

Sol^n:- Since $a_{2n} = \frac{1}{3^n}$ for $n = 0, 1, 2, \ldots$ and $a_{2n-1} = \frac{1}{3^{n+1}}$ for $n = 1, 2, 3, \ldots$

$$\lim_{n\to\infty} |a_{2n}|^{\frac{1}{2n}} = \lim_{n\to\infty} \left(\frac{1}{3^n}\right)^{\frac{1}{2n}} = \lim_{n\to\infty} \frac{1}{3^{1/2}} = \frac{1}{\sqrt{3}}$$

and $$\lim_{n\to\infty} |a_{2n-1}|^{\frac{1}{2n-1}} = \lim_{n\to\infty} \left(\frac{1}{3^{n+1}}\right)^{\frac{1}{2n-1}} = \lim_{n\to\infty} \frac{1}{3^{\frac{n+1}{2n-1}}} = \lim_{n\to\infty} \frac{1}{3^{1/2}} = \frac{1}{\sqrt{3}}.$$

Let $u_n = |a_n|^{1/n}$.

Then $u_{2n} = |a_{2n}|^{\frac{1}{2n}}$ and $u_{2n-1} = |a_{2n-1}|^{\frac{1}{2n-1}}$

The subsequence $\{u_{2n}\}$ of the sequence $\{u_n\}$ Converges to $\frac{1}{\sqrt{3}}$ and also the subsequence $\{u_{2n-1}\}$ of the sequence $\{u_n\}$ Converges to $\frac{1}{\sqrt{3}}$.

Thus the sequence $\{u_n\}$ is Convergent and Converges to $\frac{1}{\sqrt{3}}$.

Therefore, $\lim_{n\to\infty} u_n = \frac{1}{\sqrt{3}}$ i.e., $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{\sqrt{3}}$.

This shows that the radius of Convergence of the Power series $\sum_{n=0}^{\infty} a_n x^n$ is $\sqrt{3}$.

Q3. Find the radius of Convergence of the Power series

(i) $x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \cdots\cdots$

(ii) $x + \frac{(2!)^2}{4!} x^2 + \frac{(3!)}{6!} x^3 + \cdots\cdots + \frac{(n!)^2}{(2n)!} x^n + \cdots\cdots$

(iii) $1 + 3x + \frac{3^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \cdots\cdots$

(iv) $1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4} x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} x^3 + \cdots\cdots$

(v) $1 - \frac{2^2}{3^2} x + \frac{2^2\cdot 4^2}{3^2\cdot 5^2} x^2 - \frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2} x^3 + \cdots\cdots$

(vi) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} (x+1)^n$.

(vii) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)(n+2)} (x-2)^n$.

Soln:- (i) Let $\sum_{n=0}^{\infty} a_n x^n$ be the given series. Then $a_0 = 0$, $a_n = \frac{n^n}{n!}$ $\forall n \in \mathbb{N}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \left(1+\frac{1}{n}\right)^n \text{ for } n \geq 1.$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = e.$$

Thus the radius of Convergence of the power series is $\frac{1}{e}$.

(ii) Let $\sum_{n=0}^{\infty} a_n x^n$ be the given series.

Then $a_0 = 0$, $a_1 = 1$, $a_n = \frac{(n!)^2}{(2n)!}$ for all $n \geq 2$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\{(n+1)!\}^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} = \frac{n+1}{2(2n+1)} = \frac{(1+\frac{1}{n})}{2(2+\frac{1}{n})}$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}.$$

Thus the radius of Convergence of the Power Series is 4.

(iii) Let $\sum_{n=0}^{\infty} a_n x^n$ be the given series.

Then $a_0 = 1$, $a_n = \frac{3^n}{n!}$ for $n \geq 1$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3}{n+1} \text{ for } n \geq 1.$$

Therefore, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$.

Thus the radius of Convergence of the Power series is ∞.

(iv) Let $\sum_{n=0}^{\infty} a_n x^n$ be the given series.

Then $a_0 = 1$, $a_n = \frac{1\cdot3\cdot5\cdots\cdots(2n-1)}{2\cdot4\cdot6\cdots\cdots2n}$ for $n \geq 1$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1\cdot3\cdot5\cdots\cdots(2n+1)}{2\cdot4\cdot6\cdots\cdots(2n+2)}\cdot\frac{2\cdot4\cdot6\cdots\cdots2n}{1\cdot3\cdot5\cdots(2n-1)} = \frac{2n+1}{2n+2} \text{ for } n\geq 1$$

Therefore, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

Thus the radius of Convergence of the Power series is 1.

(v) Let $\sum_{n=0}^{\infty} a_n x^n$ be the given series.

Then $a_0 = 1$, $a_n = (-1)^n\cdot\frac{2^2\cdot4^2\cdots\cdots(2n)^2}{3^2\cdot5^2\cdots\cdots(2n+1)^2}$ for $n\geq 1$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^2\cdot4^2\cdots\cdots(2n)^2(2n+2)^2}{3^2\cdot5^2\cdots\cdots(2n+1)^2(2n+3)^2}\cdot\frac{3^2\cdot5^2\cdots\cdots(2n+1)^2}{2^2\cdot4^2\cdots\cdots(2n)^2} = \frac{(2n+2)^2}{(2n+3)^2}$$

Therefore, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

Thus the radius of Convergence of the Power series is 1.

(vi) Let $\sum_{n=0}^{\infty} a_n x'^n$ be the given series, where $x' = x+1$.

Then $a_n = \frac{(-1)^{n+1}}{n+1}$ for $n = 0, 1, 2, \cdots\cdots$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{n+2}\cdot(n+1) = \frac{(1+\frac{1}{n})}{(1+\frac{2}{n})}$$

Therefore, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

Thus the radius of Convergence of the series is 1.

(vii) Let $\sum_{n=0}^{\infty} a_n x'^n$ be the given series, where $x' = x-2$.

Then $a_n = \frac{(-1)^{n+1}}{(n+1)(n+2)}$ for $n = 0, 1, 2, \cdots\cdots$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(n+3)(n+2)}\cdot\frac{(n+1)(n+2)}{1} = \frac{n+1}{n+3} = \frac{1+1/n}{1+3/n} \text{ for } n = 1, 2, 3, \cdots$$

Therefore, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

Thus the radius of Convergence of the Power series is 1.

Q4. Determine the radius of Convergence of the Power Series

$\frac{1}{3} - x + \frac{x^2}{3^2} - x^3 + \frac{x^4}{3^4} - x^5 + \cdots$

Solⁿ:- Let $\sum_{n=0}^{\infty} a_n x^n$ be the given series.

Then $a_0 = \frac{1}{3}$, $a_1 = -1$, $a_2 = \frac{1}{3^2}$, $a_3 = -1$, $a_4 = \frac{1}{3^4}$, $\cdots$

Let $\overline{A_n} = \sup\{|a_n|^{1/n}, |a_{n+1}|^{\frac{1}{n+1}}, |a_{n+2}|^{\frac{1}{n+2}}, \cdots\}$ for $n = 1, 2, 3, \cdots$

Then $\overline{A_n} = 1$ for $n = 1, 2, 3, \cdots$

Thus $\overline{\lim} |a_n|^{1/n} = 1$.

Therefore the radius of Convergence of the Power series is 1.

Theorem:- Let $\sum_{n=0}^{\infty} a_n x^n$ be a Power series with radius of Convergence $R(>0)$. Then the series is uniformly Convergent on $[-s, s]$, where $0 < s < R$.

Proof:- Let $f_n(x) = a_n x^n$, $n \geq 0$.

Since R is the radius of Convergence of the Power series, the Series is absolutely convergent for all real x satisfying $|x| < R$.

Since $0 < s < R$, the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely Convergent for all x satisfying $|x| \leq s < R$.

Therefore the series $\sum_{n=0}^{\infty} |a_n s^n|$ is Convergent.

Now $|f_n(x)| = |a_n x^n| \leq |a_n| s^n$ for all real x satisfying $|x| \leq s$.

Let $M_n = |a_n| s^n$ for all $n \in \mathbb{N}$.

Then $\sum_{n=1}^{\infty} M_n$ is a Convergent series of Positive real numbers and for all $n \in \mathbb{N}$, $|f_n(x)| \leq M_n$ for all $x \in [-s, s]$.

By Weierstrass's M-test; the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly Convergent on $[-s, s]$. Consequently, the series $\sum_{n=0}^{\infty} f_n(x)$, i.e., the Power series $\sum_{n=0}^{\infty} a_n x^n$ is Uniformly Convergent on $[-s, s]$.

Theorem:- Let $R(>0)$ be the radius of Convergence of the Power series $\sum_{n=0}^{\infty} a_n x^n$. If $[a, b]$ be any closed interval Contained in $(-R, R)$, then the series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly Convergent on $[a, b]$.

Proof :- Let $S = \max\{|a|, |b|\}$.
Then $0 < S < R$ and $[a,b] \subseteq [-S,S]$.

Let $f_n(x) = a_n x^n$, $n \geq 0$.
Since R is the radius of Convergence of the Power Series $\sum_{n=0}^{\infty} a_n x^n$, the series is absolutely Convergent for all real x satisfying $|x| < R$.
Since $0 < S < R$, the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely Convergent for all x satisfying $|x| \leq S < R$.
Therefore the series $\sum_{n=0}^{\infty} |a_n S^n|$ is Convergent.
Now $|f_n(x)| = |a_n x^n| \leq |a_n| S^n$ for all real x satisfying $|x| \leq S$.
Let $M_n = |a_n| S^n$ for all $n \in N$.
Then $\sum_{n=1}^{\infty} M_n$ is a Convergent series of Positive real numbers and for all $n \in N$, $|f_n(x)| \leq M_n$ for all $x \in [-S,S]$.
By Weierstrass's M-test, the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly Convergent on $[-S,S]$. Consequently, the series $\sum_{n=0}^{\infty} f_n(x)$, i.e., the Power series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly Convergent on $[-S,S]$.
Since the Power series is uniformly Convergent on $[-S,S]$ and $[a,b] \subseteq [-S,S]$, the Power series is uniformly Convergent on $[a,b]$.

Corollary :- Let $R(>0)$ be the radius of Convergence of the Power series $\sum_{n=0}^{\infty} a_n x^n$. Then the Power series is uniformly Convergent on $[-R+\epsilon, R-\epsilon]$ where ϵ is an arbitrary small Positive number satisfying $R-\epsilon > 0$.

Proof :- $R-\epsilon > 0$. Let $S = R-\epsilon$. Then $0 < S < R$ and therefore the Power series is uniformly Convergent on $[-S,S]$, i.e., on $[-R+\epsilon, R-\epsilon]$.

Theorem :- Let $\sum_{n=0}^{\infty} a_n x^n$ be a Power series with radius of Convergence $R(>0)$. Let $f(x)$ be the sum of the series on $(-R,R)$. Then f is Continuous on $(-R,R)$.

Proof:- Since R is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$, the series is uniformly convergent on $[-R+\delta, R-\delta]$ where δ is an arbitrary small positive number satisfying $R-\delta>0$.

Let $f_n(x) = a_n x^n$, $n \geq 0$.

Let $S_n(x) = f_0(x) + f_1(x) + \cdots\cdots + f_n(x)$, $n \geq 1$.

since the series is uniformly convergent on $[-R+\delta, R-\delta]$ to the function f, the sequence $\{S_n\}$ is uniformly convergent on $[-R+\delta, R-\delta]$.

Let $c \in [-R+\delta, R-\delta]$.

Let us choose $\epsilon>0$. There exists a natural number K such that for all $x \in [-R+\delta, R-\delta]$, $|S_n(x) - f(x)| < \epsilon/3$ for all $n \geq K$.

Hence for all $x \in [-R+\delta, R-\delta]$, $|S_K(x) - f(x)| < \epsilon/3$

Therefore $|S_K(c) - f(c)| < \epsilon/3$.

Since each f_n is continuous at c, S_n is continuous at c for all $n \geq 1$.

Therefore there exists a positive δ' such that

$|S_K(x) - S_K(c)| < \epsilon/3$ for all $x \in N(c,\delta') \cap [-R+\delta, R-\delta]$. We have

$$\begin{aligned}|f(x)-f(c)| &= |f(x) - S_K(x) + S_K(x) - S_K(c) + S_K(c) - f(c)| \\ &\leq |S_K(x) - f(x)| + |S_K(x) - S_K(c)| + |S_K(c) - f(c)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \text{ for all } x \in N(c,\delta') \cap [-R+\delta, R-\delta].\end{aligned}$$

This shows that f is continuous at c.

Since c is arbitrary, f is continuous on $[-R+\delta, R-\delta]$.

since δ is arbitrary, f is continuous on $(-R, R)$.

Note:- A power series with radius of convergence $R(>0)$ has a continuous sum function on the interval of convergence $(-R, R)$.

Lemma:- Let $\{u_n\}$ be a bounded sequence where $u_n \geq 0$ for all $n \in N$ and let $v_n \geq 0$ for all $n \in N$ and $\lim_{n\to\infty} v_n = v$. Then $\overline{\lim}(u_n v_n) = v.\overline{\lim}\, u_n$.

Lemma:- Let $u_n > 0$ for all $n \in N$ and $\{u_n\}$ be a bounded sequence such that $\overline{\lim}\, u_n > 0$. Let $v_n > 0$ for all $n \in N$ and $\lim_{n\to\infty} v_n = v > 0$. Then $\overline{\lim}(u_n)^{v_n} = (\overline{\lim}\, u_n)^{v}$.

Theorem:- A Power series can be integrated term-by-term on any closed and bounded interval contained with in the interval of Convergence.

Proof:- Let $\sum_{n=0}^{\infty} a_n x^n$ be a Power series with radius of Convergence R and $f(x)$ be the sum of the series on $(-R,R)$.

Let $[a,b]$ be a closed and bounded interval such that $[a,b] \subset (-R,R)$.

Now we have to prove that, $\int_a^b a_0\,dx + \int_a^b a_1 x\,dx + \int_a^b a_2 x^2\,dx + \cdots = \int_a^b f(x)\,dx$.

Let $S_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, $n \in \mathbb{N}$.

Since R is the radius of Convergence of the Power series $\sum_{n=0}^{\infty} a_n x^n$ and the closed and bounded interval $[a,b] \subset (-R,R)$, the series is uniformly Convergent on $[a,b]$ to the sum function f. Therefore the sequence $\{S_n\}$ is uniformly Convergent on $[a,b]$ to f.

Let us choose $\epsilon > 0$. Then there exists a natural number K such that for all $x \in [a,b]$, $|S_n(x) - f(x)| < \frac{\epsilon}{b-a}$ for all $n \geq K$.

Since each term of the Power series is integrable on $[a,b]$, f is also integrable on $[a,b]$.

Now
$$\left|\int_a^b S_n(x)\,dx - \int_a^b f(x)\,dx\right| = \left|\int_a^b \{S_n(x) - f(x)\}\,dx\right|$$
$$\leq \int_a^b |S_n(x) - f(x)|\,dx$$
$$< \frac{\epsilon}{b-a}\int_a^b dx$$
$$= \epsilon \text{ for all } n \geq K.$$

it follows that, $\lim_{n\to\infty} \int_a^b S_n(x)\,dx = \int_a^b f(x)\,dx$.

i.e., $\int_a^b a_0\,dx + \int_a^b a_1 x\,dx + \int_a^b a_2 x^2\,dx + \cdots\cdots = \int_a^b f(x)\,dx$.

Note:- For any x satisfying $|x| < R$, the series is uniformly Convergent on $[0,x]$ or $[x,0]$ and

$$\int_0^x a_0\,dx + \int_0^x a_1 x\,dx + \int_0^x a_2 x^2\,dx + \cdots\cdots = \int_0^x f(x)\,dx.$$

$$\text{or, } a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \cdots + \frac{a_n x^{n+1}}{n+1} + \cdots = \int_0^x f(x)\,dx.$$

The Convergence of the left hand series (obtained by term-by-term integration) to $\int_0^x f(x)\,dx$ is established by the theorem.

Theorem:- Let $R(>0)$ be the radius of convergence of the power series $a_0+a_1x+a_2x^2+\cdots\cdots$, then the radius of convergence of the power series $a_0x+\frac{a_1x^2}{2}+\frac{a_2x^3}{3}+\cdots\cdots+\frac{a_nx^{n+1}}{n+1}+\cdots$, obtained by term-by-term integration, is also R.

Proof:- Since R is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_nx^n$, $\frac{1}{R}=\overline{\lim}|a_n|^{1/n}$.

Let R' be the radius of convergence of the power series $a_0x+\frac{a_1}{2}x^2+\frac{a_2}{3}x^3+\cdots\cdots+\frac{a_{n-1}}{n}x^n+\cdots\cdots$.

Then $\frac{1}{R'}=\overline{\lim}\left|\frac{a_{n-1}}{n}\right|^{\frac{1}{n}}$.

Now $\frac{1}{R'}=\overline{\lim}\left|\frac{a_{n-1}}{n}\right|^{1/n}=\overline{\lim}\frac{\left\{|a_{n-1}|^{\frac{1}{n-1}}\right\}^{\frac{n-1}{n}}}{n^{1/n}}=\overline{\lim}(u_nv_n)$, where $u_n=\frac{1}{n^{1/n}}$

and $v_n=\left\{|a_{n-1}|^{\frac{1}{n-1}}\right\}^{\frac{n-1}{n}}$.

As $\overline{\lim}|a_n|^{1/n}=\frac{1}{R}$, we have $\overline{\lim}|a_{n-1}|^{\frac{1}{n-1}}=\frac{1}{R}$.

Since $\overline{\lim}|a_{n-1}|^{\frac{1}{n-1}}=\frac{1}{R}$ and $\lim_{n\to\infty}\frac{n-1}{n}=1$, it follows that $\overline{\lim}\, v_n=\frac{1}{R}$.

As $\lim_{n\to\infty}n^{1/n}=1$, we have $\lim_{n\to\infty}u_n=1$.

Since $\lim_{n\to\infty}u_n=1$ and $\overline{\lim}\, v_n=\frac{1}{R}$, we have $\overline{\lim}(u_nv_n)=\lim_{n\to\infty}u_n\cdot\overline{\lim}\, v_n$
$=\frac{1}{R}$.

Therefore, $\frac{1}{R'}=\frac{1}{R}$, i.e., $R'=R$.

Note:- It follows that the series obtained by integrating the power series $\sum a_nx^n$ term-by-term is also uniformly convergent on any closed and bounded sub-interval contained in the interval of convergence.

Q5. A function f is defined on $(-\frac{1}{3},\frac{1}{3})$ by

$$f(x)=1+2\cdot3x+3\cdot3^2x^2+\cdots\cdots+n\cdot3^{n-1}x^{n-1}+\cdots\cdots$$

Show that f is continuous on $(-\frac{1}{3},\frac{1}{3})$. Evaluate $\int_0^{1/4}f(x)\,dx$.

Soln:- Let $\sum_{n=0}^{\infty}a_nx^n$ be the given series.

Then $a_0=1$, $a_n=(n+1)3^n$ for $n\geq1$.

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{3(n+2)}{n+1}=3$.

The radius of convergence of the power series is $\frac{1}{3}$.

Therefore f is Continuous on $(-\frac{1}{3}, \frac{1}{3})$.

The series can be integrated term-by-term on any closed interval Contained within $(-\frac{1}{3}, \frac{1}{3})$. $[0, \frac{1}{4}] \subset (-\frac{1}{3}, \frac{1}{3})$.

Therefore $\int_0^{1/4} f(x)\,dx$

$$= \int_0^{1/4} dx + \int_0^{1/4} 2\cdot 3x\,dx + \int_0^{1/4} 3\cdot 3^2 x^2\,dx + \cdots + \int_0^{1/4} n\cdot 3^{n-1} x^{n-1}\,dx + \cdots$$

$$= \frac{1}{4} + \frac{1}{4}\left(\frac{3}{4}\right) + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \cdots$$

$$= \frac{1}{4} \cdot \frac{1}{1-\frac{3}{4}} = 1.$$

Theorem:- Let $R(>0)$ be the radius of Convergence of the Power series $a_0 + a_1 x + a_2 x^2 + \cdots$. Then the radius of Convergence of the Power series $a_1 + 2a_2 x + 3a_3 x^2 + \cdots$ obtained by term-by-term differentiation, is also R.

Proof:- Since R is the radius of Convergence of the Power series $a_0 + a_1 x + a_2 x^2 + \cdots$, $\frac{1}{R} = \overline{\lim} |a_n|^{1/n}$.

Let R' be the radius of Convergence of the Power series $a_1 + 2a_2 x + 3a_3 x^2 + \cdots$, Then $\frac{1}{R'} = \overline{\lim} \{(n+1)|a_{n+1}|\}^{1/n}$.

$$\therefore \frac{1}{R'} = \overline{\lim} \{(n+1)|a_{n+1}|\}^{1/n}$$

$$= \overline{\lim} \left[\{(n+1)^{\frac{1}{n+1}}\}^{\frac{n+1}{n}} \{|a_{n+1}|^{\frac{1}{n+1}}\}^{\frac{n+1}{n}} \right]$$

$$= \overline{\lim} (u_n v_n) \text{ where } u_n = \{(n+1)^{\frac{1}{n+1}}\}^{\frac{n+1}{n}}, \text{ and } v_n = \{|a_{n+1}|^{\frac{1}{n+1}}\}^{\frac{n+1}{n}}.$$

As $\overline{\lim} |a_n|^{1/n} = \frac{1}{R}$, we have $\overline{\lim} |a_{n+1}|^{\frac{1}{n+1}} = \frac{1}{R}$.

Since $\overline{\lim} |a_{n+1}|^{\frac{1}{n+1}} = \frac{1}{R}$ and $\lim_{n\to\infty} \frac{n+1}{n} = 1$, it follows that $\overline{\lim}\, v_n = \frac{1}{R}$.

As $\lim_{n\to\infty} n^{1/n} = 1$, we have $\lim_{n\to\infty} (n+1)^{\frac{1}{n+1}} = 1$, it follows that $\lim_{n\to\infty} u_n = 1$.

Since $\lim_{n\to\infty} u_n = 1$ and $\overline{\lim}\, v_n = \frac{1}{R}$, we have $\overline{\lim}(u_n v_n) = \lim_{n\to\infty} u_n \cdot \overline{\lim}\, v_n$

$$= \frac{1}{R}.$$

Therefore, $\frac{1}{R'} = \frac{1}{R}$ i.e., $R' = R$.

Theorem:- A Power series can be differentiated term-by-term within the interval of Convergence. V.H-04,

Proof:- Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of Convergence $R(>0)$.

Then $\frac{1}{R} = \overline{\lim}\, |a_n|^{1/n}$.

Let $f(x)$ be the sum of the Power series on $(-R,R)$.

Now we have to Prove that

$$\frac{d}{dx}(a_0) + \frac{d}{dx}(a_1 x) + \frac{d}{dx}(a_2 x^2) + \cdots\cdots = \frac{d}{dx}\{f(x)\} \text{ on } (-R,R).$$

Differentiating the series $\sum_{n=0}^{\infty} a_n x^n$ term-by-term, we obtain the series $a_1 + 2a_2 x + 3a_3 x^2 + \cdots\cdots$

Let R' be the radius of Convergence of this Power series.

Then $\frac{1}{R'} = \overline{\lim}\, \{(n+1)|a_{n+1}|\}^{\frac{1}{n}}$.

Since
$$\overline{\lim}\, \{(n+1)|a_{n+1}|\}^{\frac{1}{n}} = \overline{\lim}\left[\{(n+1)^{\frac{1}{n+1}}\}^{\frac{n+1}{n}} \cdot \{|a_{n+1}|^{\frac{1}{n+1}}\}^{\frac{n+1}{n}}\right]$$
$$= \lim_{n\to\infty}\{(n+1)^{\frac{1}{n+1}}\}^{\frac{n+1}{n}} \cdot \overline{\lim}\,\{|a_{n+1}|^{\frac{1}{n+1}}\}^{\frac{n+1}{n}}$$
$$= 1 \cdot \overline{\lim}\,\{|a_{n+1}|^{\frac{1}{n+1}}\}, \text{ since } \lim_{n\to\infty} n^{1/n} = 1 \text{ and } \lim_{n\to\infty}\frac{n+1}{n} = 1.$$
$$= \overline{\lim}\,(|a_n|^{\frac{1}{n}}).$$

Therefore, $\frac{1}{R'} = \frac{1}{R}$ i,e., $R' = R$

Since R is the radius of Convergence of both the series, both of these are uniformly Convergent on $[-R+\epsilon, R-\epsilon]$ for any Positive ϵ satisfying $R-\epsilon > 0$.

Let $f(x)$ be the sum of the series $a_0 + a_1 x + a_2 x^2 + \cdots$ on $[-R+\epsilon, R-\epsilon]$.

Then $\frac{d}{dx}(a_0) + \frac{d}{dx}(a_1 x) + \frac{d}{dx}(a_2 x^2) + \cdots\cdots = \frac{d}{dx}\{f(x)\}$ on $[-R+\epsilon, R-\epsilon]$.

Since ϵ is arbitrary, it follows that f is differentiable at each Point of $(-R,R)$ and

$$\frac{d}{dx}(a_0) + \frac{d}{dx}(a_1 x) + \frac{d}{dx}(a_2 x^2) + \cdots\cdots = \frac{d}{dx}\{f(x)\} \text{ on } (-R,R).$$

<u>**Theorem**</u> :- Let $\sum_{n=0}^{\infty} a_n x^n$ be a Power series with radius of Convergence $R(>0)$ and $f(x)$ be the sum of the Series on $(-R,R)$. Then $f^K(0) = K!\, a_K$ $(K = 0,1,2,\cdots\cdots)$.

<u>**Proof**</u> :- $a_0 + a_1 x + a_2 x^2 + \cdots\cdots = f(x)$ on $(-R,R)$ ------(i)

Therefore $a_0 = f(0)$.

Differentiating the series (i) term-by-term, we have

$a_1+2a_2x+3a_3x^2+ \cdots\cdots = f'(x)$ on $(-R,R)$ ---- (ii)

Therefore, $a_1=f'(0)$.

Differentiating the series (ii) term-by-term, we have

$1\cdot 2a_2+3\cdot 2a_3x+4\cdot 3a_4x^2+\cdots\cdots = f''(x)$ on $(-R,R)$ -----(iii)

Therefore, $2!\,a_2=f''(0)$.

Differentiating the series (iii) term-by-term, we have

$1\cdot 2\cdot 3a_3+2\cdot 3\cdot 4a_4x+3\cdot 4\cdot 5a_5x^2+\cdots\cdots = f'''(x)$ on $(-R,R)$.

Therefore $3!\,a_3=f'''(0)$.

Proceeding similarly, $K!\,a_K=f^K(0)$ for $K=0,1,2,\cdots\cdots$.

Note:- The Power series takes the form $\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n$, the coefficients depending on the values at the origin of the sum function f and its successive derivatives.

Theorem:-(Abel)-

Let $\sum_{n=0}^{\infty}a_nx^n$ be a Power series with radius of convergence $R(>0)$. If the series converges at the end Point R of the interval of convergence $(-R,R)$, then the series is uniformly convergent on the closed interval $[0,R]$.

Proof:- The series $\sum_{n=0}^{\infty}a_nR^n$ is convergent.

Let us choose $\epsilon>0$. Then there exists a natural number K such that

$|a_{n+1}R^{n+1}+a_{n+2}R^{n+2}+\cdots\cdots+a_{n+p}R^{n+p}|<\epsilon$ for all $n\geq K$ and $p=1,2,\cdots$

Let $S_{n,1}=a_{n+1}R^{n+1}$.

$S_{n,2}=a_{n+1}R^{n+1}+a_{n+2}R^{n+2}$.

$\cdots\cdots\cdots$

$S_{n,p}=a_{n+1}R^{n+1}+a_{n+2}R^{n+2}+\cdots\cdots+a_{n+p}R^{n+p}$.

$\cdots\cdots\cdots$

Then $|S_{n,p}|<\epsilon$ for all $n\geq K$ and $p=1,2,3,\cdots\cdots$

$|a_{n+1}x^{n+1}+a_{n+2}x^{n+2}+\cdots\cdots+a_{n+p}x^{n+p}|$

$=\left|a_{n+1}R^{n+1}\left(\frac{x}{R}\right)^{n+1}+a_{n+2}R^{n+2}\left(\frac{x}{R}\right)^{n+2}+\cdots\cdots+a_{n+p}R^{n+p}\left(\frac{x}{R}\right)^{n+p}\right|$

$=\left|S_{n,1}\left(\frac{x}{R}\right)^{n+1}+(S_{n,2}-S_{n,1})\left(\frac{x}{R}\right)^{n+2}+\cdots\cdots+(S_{n,p}-S_{n,p-1})\left(\frac{x}{R}\right)^{n+p}\right|$

$$= \left| S_{n,1}\left\{\left(\tfrac{x}{R}\right)^{n+1} - \left(\tfrac{x}{R}\right)^{n+2}\right\} + S_{n,2}\left\{\left(\tfrac{x}{R}\right)^{n+2} - \left(\tfrac{x}{R}\right)^{n+3}\right\} + \cdots + S_{n,p-1}\left\{\left(\tfrac{x}{R}\right)^{n+p-1} - \left(\tfrac{x}{R}\right)^{n+p}\right\} + S_{n,p}\left(\tfrac{x}{R}\right)^{n+p} \right|$$

$$\leq |S_{n,1}|\left|\left(\tfrac{x}{R}\right)^{n+1} - \left(\tfrac{x}{R}\right)^{n+2}\right| + |S_{n,2}|\left|\left(\tfrac{x}{R}\right)^{n+2} - \left(\tfrac{x}{R}\right)^{n+3}\right| + \cdots + |S_{n,p-1}|\left|\left(\tfrac{x}{R}\right)^{n+p-1} - \left(\tfrac{x}{R}\right)^{n+p}\right| + |S_{n,p}|\left|\left(\tfrac{x}{R}\right)^{n+p}\right|.$$

$$= |S_{n,1}|\left\{\left(\tfrac{x}{R}\right)^{n+1} - \left(\tfrac{x}{R}\right)^{n+2}\right\} + |S_{n,2}|\left\{\left(\tfrac{x}{R}\right)^{n+2} - \left(\tfrac{x}{R}\right)^{n+3}\right\} + \cdots + |S_{n,p-1}|\left\{\left(\tfrac{x}{R}\right)^{n+p-1} - \left(\tfrac{x}{R}\right)^{n+p}\right\} + |S_{n,p}|\left\{\left(\tfrac{x}{R}\right)^{n+p}\right\}.$$

[Since for all $x \in [0,R]$, $0 \leq \left(\frac{x}{R}\right)^{n+p} \leq \left(\frac{x}{R}\right)^{n+p-1} \leq \cdots \leq \left(\frac{x}{R}\right)^{n+1} \leq 1$]

$$< \epsilon\left\{\left(\tfrac{x}{R}\right)^{n+1} - \left(\tfrac{x}{R}\right)^{n+2}\right\} + \epsilon\left\{\left(\tfrac{x}{R}\right)^{n+2} - \left(\tfrac{x}{R}\right)^{n+3}\right\} + \cdots + \epsilon\left\{\left(\tfrac{x}{R}\right)^{n+p-1} - \left(\tfrac{x}{R}\right)^{n+p}\right\} + \epsilon\left\{\left(\tfrac{x}{R}\right)^{n+p}\right\}.$$

$$= \epsilon \cdot \left(\tfrac{x}{R}\right)^{n+1}.$$

Therefore for all $x \in [0,R]$, $|a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots + a_{n+p}x^{n+p}| < \epsilon$ for all $n \geq K$ and $p = 1,2,3,\ldots$.

This proves that $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[0,R]$.

Theorem (Abel):- Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R(>0)$. If the series converges at the end point $-R$ of the interval of convergence $(-R,R)$, then the series is uniformly convergent on the closed interval $[-R,0]$.

Theorem:- Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence 1. If $\sum_{n=0}^{\infty} a_n$ be convergent then the series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[0,1]$.

Proof:- Let us choose $\epsilon > 0$.

Since the series $\sum_{n=0}^{\infty} a_n$ is convergent, there exists a natural number K such that $|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \epsilon$ for all $n \geq K$ and $p = 1,2,3,\ldots$

Let $S_{n,1} = a_{n+1}$,

$S_{n,2} = a_{n+1} + a_{n+2}$,

$\cdots\cdots$

$S_{n,p} = a_{n+1} + a_{n+2} + \cdots + a_{n+p}$,

$\cdots\cdots$

Then $|S_{n,p}| < \epsilon$ for all $n \geq K$ and $p = 1,2,3,\ldots$

$$\left|a_{n+1}x^{n+1}+a_{n+2}x^{n+2}+\cdots\cdots+a_{n+p}x^{n+p}\right|$$

$$=\left|S_{n,1}x^{n+1}+(S_{n,2}-S_{n,1})x^{n+2}+\cdots\cdots+(S_{n,p}-S_{n,p-1})x^{n+p}\right|.$$

$$=\left|S_{n,1}\{x^{n+1}-x^{n+2}\}+S_{n,2}\{x^{n+2}-x^{n+3}\}+\cdots+S_{n,p-1}\{x^{n+p-1}-x^{n+p}\}+S_{n,p}x^{n+p}\right|.$$

$$\leq |S_{n,1}||x^{n+1}-x^{n+2}|+|S_{n,2}||x^{n+2}-x^{n+3}|+\cdots+|S_{n,p-1}||x^{n+p-1}-x^{n+p}|+|S_{n,p}||x^{n+p}|.$$

$$=|S_{n,1}|\{x^{n+1}-x^{n+2}\}+|S_{n,2}|\{x^{n+2}-x^{n+3}\}+\cdots\cdots+|S_{n,p-1}|\{x^{n+p-1}-x^{n+p}\}+|S_{n,p}|x^{n+p},$$ for all $x\in[0,1]$

$<\epsilon\cdot x^{n+1}$ for all $n\geq K$ and $p=1,2,3,\cdots\cdots$

Therefore for all $x\in[0,1]$, $|a_{n+1}x^{n+1}+a_{n+2}x^{n+2}+\cdots+a_{n+p}x^{n+p}|<\epsilon$ for all $n\geq K$ and $p=1,2,3,\cdots\cdots$

This proves that $\sum_{n=0}^{\infty}a_nx^n$ is uniformly convergent on $[0,1]$.

Theorem :- Abel's theorem (Limit form) :-

Let $\sum_{n=0}^{\infty}a_nx^n$ be a power series with radius of convergence R and let the sum of the series be $f(x)$ on $(-R,R)$. If the series $\sum_{n=0}^{\infty}a_nR^n$ be convergent then $\sum_{n=0}^{\infty}a_nR^n=\lim_{x\to R^-}f(x)$.

Proof :- Since R is the radius of convergence of the power series and $\sum_{n=0}^{\infty}a_nR^n$ is convergent, the series is uniformly convergent on $[0,R]$. Let $\phi(x)$ be the sum of the series on $[0,R]$.

Since each term of the series is continuous on $[0,R]$, the sum function ϕ is also continuous on $[0,R]$. Also $\phi(x)=f(x)$ on $[0,R)$.

Since ϕ is continuous at R, $\phi(R)=\lim_{x\to R^-}\phi(x)=\lim_{x\to R^-}f(x)$.

Therefore $\sum_{n=0}^{\infty}a_nR^n=\lim_{x\to R^-}f(x)$.

Corollary :- Let $\sum_{n=0}^{\infty}a_nx^n$ be a power series with radius of convergence 1 and let the sum of the series be $f(x)$ on $(-1,1)$. Then

(i) if the series $\sum a_n$ be convergent, then $\sum a_n=\lim_{x\to 1^-}f(x)$;

(ii) if the series $\sum(-1)^n a_n$ be convergent, then $\sum(-1)^n a_n = \lim_{x \to -1^+} f(x)$.

Note:- The converse of Abel's theorem is not true. For a power series $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence R, $\lim_{x \to R^-} f(x)$ may exist, yet the series $\sum_{n=0}^{\infty} a_n x^n$ may not converge at R.

For example, the sum of the series $1 - x + x^2 - x^3 + \cdots$ is $\frac{1}{1+x}$ on $(-1,1)$, 1 being the radius of convergence of the series.
$\lim_{x \to 1^-} \frac{1}{1+x} = \frac{1}{2}$. but the series $1 - x + x^2 - x^3 + \cdots$ is not convergent at $x = 1$.

Theorem (Uniqueness theorem):-

If two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge on the same interval $(-R, R)$, $R > 0$, to the same function f, then
$a_n = b_n$ for ~~[illegible]~~ $n = 0, 1, 2, \cdots$ [V.H-99.

Proof:- By the given condition,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots \text{ on } (-R, R)$$
and $$f(x) = b_0 + b_1 x + b_2 x^2 + \cdots \text{ on } (-R, R).$$

At $x = 0$, $f(0) = a_0 = b_0$.

Differentiating both the series term-by-term, we have

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \text{ on } (-R, R)$$
and $$f'(x) = b_1 + 2b_2 x + 3b_3 x^2 + \cdots \text{ on } (-R, R).$$

At $x = 0$, $f'(0) = a_1 = b_1$.

Differentiating again, we have by similar arguments
$$f''(0) = a_2 = b_2.$$

Proceeding similarly, we have $a_n = b_n$ for $n = 0, 1, 2, \cdots$.

Theorem:- If R_1, R_2 be the radii of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ respectively and $\sum_{n=0}^{\infty} a_n x^n = f(x)$ for $|x| < R_1$, $\sum_{n=0}^{\infty} b_n x^n = g(x)$ for $|x| < R_2$, then the radius of convergence of the series $\sum_{n=0}^{\infty} (a_n + b_n) x^n$ is $R = \min\{R_1, R_2\}$ and the sum of the series is $f(x) + g(x)$ on $(-R, R)$.

Abel's theorem:- If the series $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, $\sum_{n=0}^{\infty} c_n$ converge to A, B, C respectively and if $c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$, Then $C = AB$.

Q6. Let $f(x)$ be the sum of the power series $\sum_{n=0}^{\infty} a_n x^n$ on $(-R, R)$ for some $R (>0)$. If $f(x) = f(-x)$ for all $x \in (-R, R)$, show that $a_n = 0$ for all odd n.

Solⁿ:- $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots = f(x)$ for all $x \in (-R, R)$

Therefore $a_0 - a_1 x + a_2 x^2 - a_3 x^3 + \cdots = f(-x)$ for all $x \in (-R, R)$.

As $f(x) = f(-x)$, both the power series $a_0 + a_1 x + a_2 x^2 + \cdots$ and $a_0 - a_1 x + a_2 x^2 - a_3 x^3 + \cdots$ have the same sum $f(x)$ on $(-R, R)$.

By uniqueness theorem, $a_1 = -a_1$, $a_3 = -a_3$, $a_5 = -a_5$, ------

Hence $a_n = 0$ for all odd n.

Q7. Assuming the power series expansion for $\frac{1}{\sqrt{1-x^2}}$ as

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1\cdot3}{2\cdot4}x^4 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^6 + \cdots$$

Obtain the power series expansion for $\sin^{-1}x$. <u>V.H-04,</u>

Deduce that $1 + \frac{1}{2\cdot3} + \frac{1\cdot3}{2\cdot4\cdot5} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot7} + \cdots = \frac{\pi}{2}$.

Solⁿ:- Let $x^2 = y$. The series becomes $1 + \frac{1}{2}y + \frac{1\cdot3}{2\cdot4}y^2 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}y^3 + \cdots$

Let $\sum_{n=0}^{\infty} a_n y^n$ be the series. Then $a_0 = 1$, $a_n = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}$ for $n \geq 1$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2n+1}{2n+2} = 1.$$

Hence the interval of convergence of the series is $\{y \in \mathbb{R} : -1 < y < 1\}$.
It follows that the interval of convergence of the given series is $\{x \in \mathbb{R} : -1 < x < 1\}$.

Integrating term-by-term on $[0, x]$ where $|x| < 1$,

$$x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot3}{2\cdot4}\frac{x^5}{5} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^7}{7} + \cdots = \sin^{-1}x \text{ on } (-1, 1).$$

At $x = 1$, the series becomes $1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5} + \cdots = \sum_{n=0}^{\infty} u_n$ (1)

Then $u_0 = 1$, $u_n = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}\cdot\frac{1}{2n+1}$.

$$\therefore \frac{u_n}{u_{n+1}} = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}\frac{1}{(2n+1)} \times \frac{2\cdot4\cdot6\cdots(2n)(2n+2)}{1\cdot3\cdot5\cdots(2n-1)(2n+1)}(2n+3)$$

$$= \frac{(2n+2)(2n+3)}{(2n+1)^2}$$

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left\{\frac{4n^2+10n+6-4n^2-4n-1}{(2n+1)^2}\right\} = \frac{n(6n+5)}{(2n+1)^2}$$

Therefore, $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{6}{4} = \frac{3}{2} > 1$.

Hence by Raabe's test, the series (1) is Convergent.

By Abel's theorem the sum of the series at $x=1$ is $\sin^{-1}(1)$.

At $x=-1$, the series becomes $-1-\frac{1}{2}\cdot\frac{1}{3}-\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5}-\cdots$

This is also Convergent.

By Abel's theorem the sum of the series at $x=-1$ is $\sin^{-1}(-1)$.

Hence $\sin^{-1}x = x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot3}{2\cdot4}\frac{x^5}{5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^7}{7}+\cdots$ for $-1\le x\le 1$.

and $1+\frac{1}{2\cdot3}+\frac{1\cdot3}{2\cdot4\cdot5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot7}+\cdots = \frac{\pi}{2}$.

Q8. Assuming the expansion $\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$ for $-1\le x\le 1$.

Prove that $\int_0^1 \frac{\log(1+x)}{x}dx = 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$

Solⁿ:- Let us Consider the series $1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\cdots$ (i)

The radius of Convergence of the series is 1. Let $f(x)$ be the sum of the series on $-1<x<1$.

Then $f(x) = \frac{\log(1+x)}{x}$, for $0<|x|<1$

$= 1$, for $x=0$.

At $x=1$, the series is convergent. By Abel's theorem, the sum of the series at $x=1$ is $\lim_{x\to1-} f(x) = \log 2$.

At $x=-1$, the series is divergent.

Hence $1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\cdots = f(x)$ for $-1<x<1$,

where $f(x) = \frac{\log(1+x)}{x}$, for $-1<x\le1$, $x\neq0$

$=1$, $x=0$.

The series (i) is uniformly Convergent on $[0,1]$. Integrating term-by-term on $[0,1]$, we have

$$\int_0^1 f(x)\,dx = \int_0^1 dx-\int_0^1\frac{x}{2}dx+\int_0^1\frac{x^2}{3}dx-\int_0^1\frac{x^3}{4}dx+\cdots$$

or, $$\int_0^1 \frac{\log(1+x)}{x}dx = 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$$

Q9. Assuming the expansion $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$ for $-1 \le x < 1$. Prove that $\int_0^1 \log(1-x)\,dx = -1$.

Sol^n:- The radius of Convergence of the series $-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$ is 1.

Integrating term-by-term on $[0,x]$ where $|x|<1$, we have

$-\frac{x^2}{2} - \frac{x^3}{2\cdot3} - \frac{x^4}{3\cdot4} - \cdots = \int_0^x \log(1-x)\,dx$ on $(-1,1)$

At $x=1$, the series becomes $-\frac{1}{1\cdot2} - \frac{1}{2\cdot3} - \frac{1}{3\cdot4} - \cdots - \frac{1}{n(n+1)} - \cdots$

Let $S_n = -\frac{1}{1\cdot2} - \frac{1}{2\cdot3} - \frac{1}{3\cdot4} - \cdots - \frac{1}{n(n+1)}$.

then $S_n = -\left[1 - \frac{1}{n+1}\right]$ and $\lim_{n\to\infty} S_n = -1$.

Therefore the series Converges to -1 at $x=1$.

By Abel's theorem, the sum of the series at $x=1$ is

$\lim_{x\to1^-} \int_0^x \log(1-x)\,dx$.

Therefore, $\lim_{x\to1^-} \int_0^x \log(1-x)\,dx = -1$.

Since $\lim_{x\to1^-} \int_0^x \log(1-x)\,dx$ exists, this limit is $\int_0^1 \log(1-x)\,dx$.

Therefore, $\int_0^1 \log(1-x)\,dx = -1$.

Q10. Find the sum of the series $\sum_{n=0}^{\infty}(2^n+3^n)x^n$, including the range of validity.

Sol^n:- Let the series be $\sum_{n=0}^{\infty} a_n x^n$. Then $a_n = 2^n + 3^n$.

$\sum_{n=0}^{\infty} 2^n x^n$ is a Power series whose radius of Convergence is $\frac{1}{2}$ and the sum of the series is $\frac{1}{1-2x}$ for $|x|<\frac{1}{2}$.

$\sum_{n=0}^{\infty} 3^n x^n$ is a Power series whose radius of Convergence is $\frac{1}{3}$ and the sum of the series is $\frac{1}{1-3x}$ for $|x|<\frac{1}{3}$.

Hence the radius of Convergence of the series $\sum_{n=0}^{\infty} a_n x^n$ is $\frac{1}{3}$ and $\sum_{n=0}^{\infty} a_n x^n = \frac{1}{1-2x} + \frac{1}{1-3x}$ for $|x|<\frac{1}{3}$.

At $x=\frac{1}{3}$ the series becomes $\sum_{n=0}^{\infty} \frac{2^n+3^n}{3^n}$.

As $\lim_{n\to\infty}\left[\left(\frac{2}{3}\right)^n+1\right]\neq 0$, the series is divergent at $x=\frac{1}{3}$.

By similar arguments, the series is divergent at $x=-\frac{1}{3}$.

Hence the sum of the series is $\frac{1}{1-2x}+\frac{1}{1-3x}$ for $-\frac{1}{3}<x<\frac{1}{3}$.

Q11. $\sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence $R(>0)$. Construct a power series $\sum_{n=0}^{\infty} b_n x^n$, other than $\sum_{n=0}^{\infty} x^n$, such that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n b_n x^n$ is also R.

V.H-05.

Soln:- Since R is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$, therefore we have, $\frac{1}{R}=\overline{\lim}\,|a_n|^{1/n}$.

Let $b_n=\frac{1}{n+1}$ for $n=0,1,2,\ldots\ldots$

Let R' be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n b_n x^n$.

Then we have, $\frac{1}{R'}=\overline{\lim}\,|a_n b_n|^{1/n}$.

Now we have to prove that $R'=R$.

Therefore, $\overline{\lim}\,|a_n b_n|^{1/n}=\overline{\lim}\,|a_n|^{1/n}\cdot\lim_{n\to\infty}\frac{1}{\left\{(n+1)^{\frac{1}{n+1}}\right\}^{\frac{n+1}{n}}}$

$=\overline{\lim}\,|a_n|^{1/n}$, since $\lim_{n\to\infty} n^{1/n}=1$.

Thus we have, $\frac{1}{R'}=\frac{1}{R}$ i.e., $R'=R$.

Q12. $\sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence $R(>0)$. Construct a power series $\sum_{n=0}^{\infty} b_n x^n$ other than $\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$, such that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n b_n x^n$ is also $2R$.

Soln:- Since R is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$, therefore we have, $\frac{1}{R}=\overline{\lim}\,|a_n|^{1/n}$.

Let $b_n=\frac{1}{2^n}$ for $n=0,2,4,\ldots\ldots$

$=\frac{1}{3^n}$ for $n=1,3,5,\ldots\ldots$

Let $\overline{B}_n=\sup\left\{|b_n|^{1/n},|b_{n+1}|^{\frac{1}{n+1}},|b_{n+2}|^{\frac{1}{n+2}},\ldots\ldots\right\}$ for $n=1,2,3,\ldots$

Then $\overline{B}_n=\frac{1}{2}$ for all $n=1,2,3,\ldots\ldots$

Therefore, $\overline{\lim}\,|b_n|^{1/n} = \frac{1}{2}$.

Let R' be the radius of convergence of the Power series $\sum_{n=0}^{\infty} a_n b_n x^n$.

Then we have, $\frac{1}{R'} = \overline{\lim}\,|a_n b_n|^{1/n} = \overline{\lim}\,|a_n|^{1/n}\,\overline{\lim}\,|b_n|^{1/n} = \frac{1}{R}\cdot\frac{1}{2}$

Therefore, $R' = 2R$.

Q13. Show that the Power series $1+2x+3x^2+4x^3+\cdots\cdots$ Converges absolutely if $-1<x<1$. Show also that the series Converges uniformly in any interval $[-K,K]$ where $0<K<1$. Hence show that $\lim_{n\to\infty} nx^{n-1} = 0$.

Sol^n:- Let $\sum_{n=0}^{\infty} a_n x^n$ be the given series.

Then $a_n = (n+1)$ for $n = 0,1,2,\cdots\cdots$

Now $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+2}{n+1}$ and therefore, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$.

It follows that the interval of Convergence of the given series is $(-1,1)$.

Therefore, the given series Converges absolutely on $(-1,1)$.

Since $0<K<1$, therefore $[-K,K]\subset(-1,1)$ and the given series Converges uniformly on $[-K,K]$.

2nd Part :- The series $\sum_{n=0}^{\infty}(n+1)x^n$ is Convergent for $-1<x<1$.

Let $u_n = (n+1)x^n$

Let us choose $\epsilon>0$. Then there exists a natural number K such that for all $x\in(-1,1)$, $|u_{n+1}+u_{n+2}+\cdots\cdots+u_{n+p}|<\epsilon$ for all $n\geq K$ and $p = 1,2,3,\cdots\cdots$

Taking $p=1$; $|u_{n+1}|<\epsilon$ for all $n\geq K$ and $x\in(-1,1)$.

That is $\lim_{n\to\infty} u_n = 0$ for all $x\in(-1,1)$.

or, $\lim_{n\to\infty} nx^{n-1} = 0$ for all $x\in(-1,1)$.

Q14. Find the radius of Convergence (R) of the Power series $\frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots\cdots$ and examine whether the series Convergent when $x = R$.

Sol^n:- Let $\sum_{n=0}^{\infty} a_n x^n$ be the given series.

Then $a_n = \frac{1\cdot 3\cdot 5 \cdots\cdots (2n-1)}{2\cdot 4\cdot 6 \cdots\cdots 2n}$, $n \geq 1$ and $a_0 = 0$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1\cdot 3\cdot 5 \cdots\cdots (2n-1)(2n+1)}{2\cdot 4\cdot 6 \cdots\cdots 2n(2n+2)} \times \frac{2\cdot 4\cdot 6 \cdots\cdots 2n}{1\cdot 3\cdot 5 \cdots\cdots (2n-1)}$$

$$= \frac{2n+1}{2n+2} \text{ for } n \geq 1$$

Therefore, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

Therefore the radius of Convergence of the series is 1.

When $x = 1$, the given series becomes,

$\frac{1}{2} + \frac{1\cdot 3}{2\cdot 4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} + \cdots\cdots = \sum_{n=1}^{\infty} u_n$ (say).

Then $u_n = \frac{1\cdot 3\cdot 5 \cdots\cdots (2n-1)}{2\cdot 4\cdot 6 \cdots\cdots 2n}$.

Therefore, $\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}$.

$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} < 1$.

By Raabe's test, the given series is divergent for $x = 1$.

Q15. Discuss the Convergence of $1 + \frac{x}{2} + \left(\frac{x}{4}\right)^2 + \left(\frac{x}{2}\right)^3 + \left(\frac{x}{4}\right)^4 + \cdots\cdots$ **V.H-07**

Sol^n:- Let $\sum_{n=0}^{\infty} a_n x^n$ be the given series.

Then $a_{2n} = \left(\frac{1}{4}\right)^{2n}$, $a_{2n-1} = \left(\frac{1}{2}\right)^{2n-1}$ for $n = 1, 2, 3, \cdots\cdots$ and $a_0 = 1$.

Let $\bar{A}_n = \sup\{|a_n|^{1/n}, |a_{n+1}|^{\frac{1}{n+1}}, |a_{n+2}|^{\frac{1}{n+2}}, \cdots\cdots\}$ for $n = 1, 2, 3, \cdots$

Then $\bar{A}_n = \frac{1}{2}$ for $n = 1, 2, 3, \cdots\cdots$ and therefore, $\overline{\lim} |a_n|^{1/n} = \frac{1}{2}$.

Thus the radius of Convergence of the given series is 2 and the series converges for $-2 < x < 2$.

At $x = \pm 2$, all odd terms have absolute value 1.

That is $\lim_{n\to\infty} a_n \neq 0$, whereby the series diverges at each end.

Q16. Find the series for $\log(1+x)$ by integration, and use Abel's theorem to show that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots = \log 2$. [V.H-00,

Sol^n:- We have $\frac{d}{dx}\log(1+x) = \frac{1}{1+x}$.

Now, $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots\cdots = \sum_{n=0}^{\infty} a_n x^n$ (say) ---- (1)

Then $a_n = (-1)^n, n = 0, 1, 2, \cdots\cdots$

$\left|\frac{a_{n+1}}{a_n}\right| = 1$ for all n.

Therefore, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

Thus the radius of Convergence of the given series is 1. It follows that the interval of Convergence of the series (1) is $\{x \in \mathbb{R} : -1 < x < 1\}$.

Integrating (1) term-by-term on $[0, x]$ where $|x| < 1$,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots \text{ on } (-1, 1).$$

At $x = 1$, the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$ which is Convergent.

By Abel's theorem the sum of the series at $x = 1$ is $\log 2$.

At $x = -1$, the series becomes $-1 - \frac{1}{2} - \frac{1}{3} - \cdots\cdots$ which is divergent.

Hence $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots$ for $-1 < x \leq 1$.

and $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots = \log 2$.

Q17: Find the sum of the power series $1 + x + x^2 + \cdots\cdots$ on its interval of Convergence. Deduce the power series expansion of $\log(1-x)$ and Use Abel's theorem to prove that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots = \log 2$.

Sol^n:- Same as the above Problem. (Try yourself).

Q18: Find the Power Series for $\tan^{-1}x$ by integration and deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots = \frac{\pi}{4}$. V.H-01,

Sol^n:- We have, $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots\cdots$ (1)

It is clear that from the ratio test the series converges for $|x| < 1$ and diverges for $|x| > 1$.

Hence the interval of convergence of the series (1) is $\{x \in \mathbb{R} : -1 < x < 1\}$.

Integrating term-by-term on $[0, x]$ where $|x| < 1$,

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\cdots \text{ on } (-1, 1) \text{ ---- (2)}$$

At $x = 1$, the series (2) becomes $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots$ whi's is Convergent by Leibnitz's test.

By Abel's theorem the sum of the series at $x = 1$ is $\tan^{-1}(1)$.

At $x=-1$, the series becomes $-1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-----$. This is also convergent.

By Abel's theorem the sum of the series at $x=-1$ is $\tan^{-1}(-1)$.

Hence $\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+-----$ for $-1\leq x\leq 1$.

and $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+---------=\frac{\pi}{4}$.

Q19. Show that $\frac{1}{2}[\log(1+x)]^2 = \frac{x^2}{2}-\frac{x^3}{3}(1+\frac{1}{2})+\frac{x^4}{4}(1+\frac{1}{2}+\frac{1}{3})-\cdots$, $-1<x\leq 1$.

Soln:- We know $\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+----$, $-1<x\leq 1$,

and $(1+x)^{-1} = 1-x+x^2-x^3+x^4---------$, $-1<x<1$.

Both the series are absolutely convergent in $(-1,1)$, therefore their Cauchy product will converge to $(1+x)^{-1}\log(1+x)$. Thus

$$(1+x)^{-1}\log(1+x) = x-x^2(1+\tfrac{1}{2})+x^3(1+\tfrac{1}{2}+\tfrac{1}{3})------,\ -1<x<1.$$

Integrating, $\frac{1}{2}[\log(1+x)]^2 = \frac{x^2}{2}-\frac{x^3}{3}(1+\frac{1}{2})+\frac{x^4}{4}(1+\frac{1}{2}+\frac{1}{3})----$,

$-1<x<1$, the constant of integration vanishes.

Since the series on the right converges at $x=1$ also, therefore by Abel's theorem, we have

$$\tfrac{1}{2}[\log(1+x)]^2 = \frac{x^2}{2}-\frac{x^3}{3}(1+\tfrac{1}{2})+\frac{x^4}{4}(1+\tfrac{1}{2}+\tfrac{1}{3})-----,\ -1<x\leq 1.$$

Q20. Assuming that $\tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}-----$ for $-1\leq x\leq 1$ and $\frac{1}{1+x^2} = 1-x^2+x^4-x^6+--------$ for $-1<x<1$, deduce that

$$\tfrac{1}{2}(\tan^{-1}x)^2 = \tfrac{1}{2}x^2-\tfrac{1}{4}(1+\tfrac{1}{3})x^4+\tfrac{1}{6}(1+\tfrac{1}{3}+\tfrac{1}{5})x^6---- \text{ for } -1\leq x\leq 1.$$

Soln:- The radius of convergence of each of the series $x-\frac{x^3}{3}+\frac{x^5}{5}----$ and $1-x^2+x^4-x^6+----$ is 1 and therefore both the series are absolutely convergent for $-1<x<1$.

Let the Cauchy product be $c_0+c_1x+c_2x^2+c_3x^3+-------$.

Then $c_0=0$, $c_1=1$, $c_2=0$, $c_3=-(1+\frac{1}{3})$, $c_4=0$, $c_5=(1+\frac{1}{3}+\frac{1}{5})$, -----

Therefore, $\frac{\tan^{-1}x}{1+x^2} = x-(1+\frac{1}{3})x^3+(1+\frac{1}{3}+\frac{1}{5})x^5-----$ for $-1<x<1$.

Integrating the series term-by-term on $[0,x]$ where $|x|<1$, we have

$$\frac{1}{2}(\tan^{-1}x)^2 = \frac{1}{2}x^2 - \frac{1}{4}\left(1+\frac{1}{3}\right)x^4 + \frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right)x^6 - \cdots \text{ for } -1<x<1.$$

At $x=\pm 1$ the series becomes $\frac{1}{2} - \frac{1}{4}\left(1+\frac{1}{3}\right) + \frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right) - \cdots$

This is an alternating series and it is convergent by Leibnitz's test.

By Abel's theorem,

$$\frac{1}{2}(\tan^{-1}x)^2 = \frac{1}{2}x^2 - \frac{1}{4}\left(1+\frac{1}{3}\right)x^4 + \frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right)x^6 - \cdots \text{ for } -1\le x\le 1.$$

Q21. Without finding the sum $f(x)$ of the series

$$1+\frac{x^2}{1!}+\frac{x^4}{2!}+\cdots+\frac{x^{2n}}{n!}+\cdots, \quad -\infty<x<\infty.$$

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Show that $f'(x) = 2x\,f(x)$ in $-\infty<x<\infty$.

Sol^n:- Let $x^2=y$. The series becomes $1+\frac{y}{1!}+\frac{y^2}{2!}+\frac{y^3}{3!}+\cdots$

Let $\sum_{n=0}^{\infty} a_n y^n$ be the series. Then $a_0=1$, $a_n=\frac{1}{n!}$, $n\ge 1$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0.$$

Hence the radius of convergence of the series is ∞. So, the interval of convergence of the series is $\{y\in\mathbb{R}: -\infty<y<\infty\}$. It follows that the interval of convergence of the given series is $\{x\in\mathbb{R}: -\infty<x<\infty\}$.

Now $f(x) = 1+\frac{x^2}{1!}+\frac{x^4}{2!}+\cdots, \quad -\infty<x<\infty.$

Since a power series can be differentiated term-by-term within the interval of convergence, therefore we have by differentiating term-by-term on $-\infty<x<\infty$,

$$f'(x) = 2x+\frac{4x^3}{2!}+\frac{6x^5}{3!}+\cdots$$

$$= 2x\left(1+\frac{x^2}{1!}+\frac{x^4}{2!}+\cdots\right)$$

$$= 2x\cdot f(x) \text{ on } (-\infty,\infty).$$