COMPLEX ANALYSIS

The Fundamental idea of extending the real number system by the introduction of complex numbers was first necessitated by the solution of algebraical equations.

There is no real number x which satisfies the polynomial equation $x^2+1=0$. To solve this eqn, Euler was the first mathematician who introduced the symbol $i=\sqrt{-1}$, called as the imaginary unit.

We can consider a complex number as having the form $a+ib$, where a & b are real numbers.

We define a complex number as an ordered pair (a,b) of real numbers a & b.

Definition: Any ordered pair of real numbers a,b written as (a,b) is called a complex numbers.

Thus, the complex number $z=a+ib$ is defined as the ordered pair (a,b) of real numbers a,b as $z=(a,b)$, where a is called real part of z and b is called imaginary part of z. and is denoted by

$$Re\,z=a \quad \text{and} \quad Im\,z=b.$$

The symbol z, which stands for any of a set of complex numbers, is called a complex variable.

NOTE-1: If $a=0$ i.e $z=ib$, then z called pure imaginary number and ib means units of imaginary number.

ii) It is not appropriate to call the expression $z=a+ib$ as an imaginary number, Call it Complex number.

iii) The Significance of '+' Sign in the expression $a+ib$ is the a real units are Super-imposed in order by imaginary units to get the Complex number $z=a+ib$.

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* Remark: The term imaginary number does not mean that such a number does not exist. The letter i is merely a symbol which denotes imaginary units just as 1 denotes the unit of real numbers.

* Equality of Complex Numbers:

Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$

NOTE: The phrases "greater than" or "less than" have no meaning in the set of complex number $\mathbb{C}$.

* Fundamental OPERATIONS:

If $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, then the laws are follows:

i) Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

ii) Subtraction: $(x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$

iii) Multiplication: $(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$

iv) Division: $\dfrac{(x_1, y_1)}{(x_2, y_2)} = \dfrac{(x_1, y_1).(x_2, -y_2)}{(x_2, y_2).(x_2, -y_2)} = \left(\dfrac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \dfrac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2} \right)$

, provided $x_2^2 + y_2^2 \neq 0$.

If $z_1, z_2, z_3 = (x_3, y_3)$ be three complex numbers then

v) Commutative law of addition: $z_1 + z_2 = z_2 + z_1$

vi) Associative law of addition: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

vii) Commutative law of multiplication: $z_1 z_2 = z_2 z_1$

viii) Associative law of multiplication: $z_1 (z_2 z_3) = (z_1 z_2) z_3$

ix) Distributive law of multiplication: $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$.

x) Additive identity: $(x, y) + (0, 0) = (x + 0, y + 0) = (x, y)$

Therefore complex number $(0, 0)$ is the additive identity and is called null or zero of the system of complex numbers.

xi) Additive inverse: $(x, y) + (-x, -y) = (x - x, y - y) = (0, 0)$

The number $(-x, -y)$ is the additive inverse of (x, y) and

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is called negative of the complex number (x,y) and we write

$$-(x,y) = (-x,-y).$$

(xii) Multiplicative identity: $(x,y)\cdot(1,0) = (x\cdot 1 - y\cdot 0, x\cdot 0 + y\cdot 1) = (x,y)$

The complex number $(1,0)$ is called multiplicative identity and also called 'one' or unity of the system of complex number.

(xiii) Multiplicative Inverse:

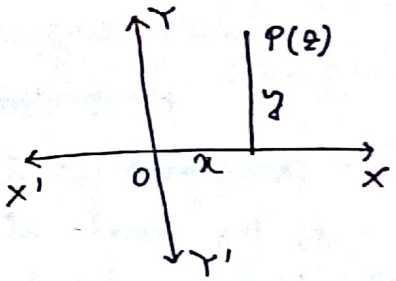
The inverse element of (x,y) is $\left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}\right)$ Since

$$(x,y)\cdot\left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}\right) = (1,0).$$

* Geometrical Representation of Complex Numbers:

Consider the complex number $z = x+iy$.

By definition, $z = (x,y)$



This form of z suggests that z can be represented by a point P whose coordinates are x and y with respect to two rectangular axes XOX' and YOY', usually called the real and imaginary axes, respectively. The complex number z is called affix of the point (x,y) which represents it.

Thus to each complex number z there corresponds one and only one point in the xy-plane and conversly to each point in this plane there exists one and only one complex number. Such a plane is called Complex plane or Gaussian plane or Argand plane.

The ~~corresponding~~ representation of complex numbers is called Argand diagram.

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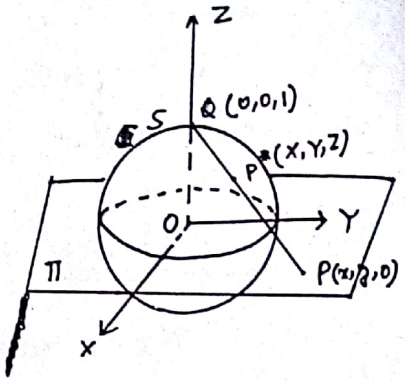
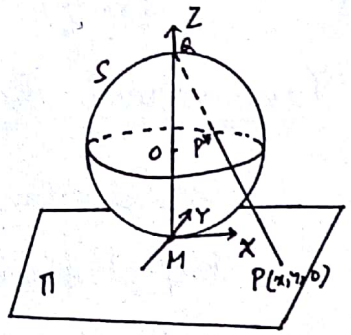
⊛ Spherical Representation of Complex Numbers: (Stereographic Projection):

Let π be the complex plane and consider a unit sphere S [radius one] tangent to π at $z=0$. i.e xy-plane.



Corresponding to any point P on π, we can construct line QP intersecting S at point P*.

Thus, each point of the complex plane π there corresponds one and only one point of the sphere S and we can represent any complex number by a point on the sphere.



For completeness we say that the point Q itself corresponds to the "point at infinity" of the plane. The set of all points of the complex plane including the point at infinity is called the entire complex plane, the entire z plane, or the extended complex plane.

The above method for mapping the plane on the to the sphere is called Stereographic Projection.

The sphere is called Riemann Sphere, in honour of Riemann who introduced this idea.

* POINT SETS:

Any finite or infinite aggregate of complex numbers is called a set of complex numbers and the corresponding points are said to constitute a point-set or simply a set, the Argand plane.

i) Neighbourhood of point z_0:

By a neighbourhood of a point z_0 in the Argand diagram, we mean a set of all points z such that $|z-z_0|<\delta$, where δ is an arbitrary small positive number, δ is called

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the reading of this nbd.

*ii) Deleted Neighbourhood of z_0: If the neighbourhood of the point z_0 defined by $0 < |z - z_0| < \delta$, then such a nbd is called the deleted nbd of z_0.

*iii) Limit Point: A point z_0 is called a limit point, cluster or accumulation of a point set S in the Argand plane z_0, if every deleted δ neighbourhood of z_0 contains points of S.

Thus, this definition implies that every nbd of z_0 contains infinite number of many points of S.

* iv) Interior, Exterior and Boundary Points:

A point z_0 is called an interior point of a set S if we can find a δ-nbd of z_0 all of whose points belongs to S.

If every δ-neighbourhood of z_0 contains points belongs to S and also points not belonging to S, then z_0 is called a boundary point.

If a point is not an interior or boundary point of a set S, it is an exterior point of S.

* v) Open Set: An open set is a set which consists only of interior points. For example, the set of points z such that $|z| < 1$ is an open set.

* vi) closed Set: A set S is said to be closed if every limit point (both inner & boundary) of S belongs to S.

For example, the set of points z s.t $|z| \leq 1$ is a closed set.

* vii) Bounded Set and Unbounded Set:

A set of points is said to be bounded if there exists a finite positive number k such that $|z| \leq k$ for all points z of the set. If there exists no such number k, the set is unbounded.

* viii) Polygonal arc: By a 'polygonal arc' we mean a continuous chain of a finite number of line segments.

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*ix) Jordan arc: A curve Γ is called a Jordan arc or a simple curve if it has no multiple points i.e if there exists some parametric representation

$$z = z(t) = x(t) + iy(t), \quad \alpha \le t \le \beta.$$

such that $t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$ i.e $z(t)$ is one-one.

The simplest example of Jordan arc is a straight-line segment.

x) Simple closed Jordan curve:

If, in a Jordan arc, the initial and terminal points coincide, i.e if there is a double point corresponding to the end points (α and β) in the interval $I: \alpha \le t \le \beta$ and there is no other multiple point on it then it is called simple closed Jordan curve or simply a closed Jordan curve.

It should be observed that a Jordan arc does not necessary posses a 'length'. If however, it does, it is called a path-segment or merely a rectifible Jordan arc.

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* Function of a Complex variable:

Let D be an arbitrary non-empty point set of the complex plane. If z be allowed to denote any point of D, z is called a complex variable or the domain of definition of z or simply the domain.

Definition: A complex variable w is said to be a function of the complex variable z, if to every value of z in a certain domain D there corresponds one or more values of w.

Thus, if w is a function of z, it is written as $w = f(z)$.

If $w = u + iv$ and $z = x + iy$, then u & v both the functions of x, y; So, we can write

$w = f(z) = u(x,y) + iv(x,y)$. , where x and y are real.

Single-valued function: A function w of the complex variable z with domain of definition D and range R is said to be single-valued or one-valued if w takes only one value in R for each value of z in D.

Multiple-valued function: If there correspond two or more values of w in R for some or all values of z in D, then w is called a multiple-valued or multi-valued or many valued function of z.

* **Example-1:** If $w = z^2$, then to each value of z there is only one value of w.

Hence $w = f(z) = z^2$ is a single-valued function of z.

* **Example-2:** If $w = z^{1/2}$, then to each value of z there are two values of w. Hence $w = f(z) = z^{1/2}$ is a multi-valued (in this case two-valued) function of z.

* **NOTE:** Whenever we speak of function we shall, Unless otherwise stated, assume Single-valued function.

* Path of a Complex variable:

In case of the function f(z) of a real variable

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the independent variable z takes values corresponding to the points lying on the x-axis. But in the case of the function f(z) of a complex variable the independent variable z takes values lying in a plane (called z-plane), so that path of z is a polygonal arc joining the initial and the final point z. This path may be a straight line or a curve.

So path of the complex variable z is either a straight line or a curve.

* Point SETS:

Any collection of points in the complex plane is called a (two-dimensional) point set and each point is called a member.

* Inverse Functions:

If $w = f(z)$ then we can also consider z as a function of w, written $z = g(w) = f^{-1}(w)$. The function f^{-1} is often called the inverse function corresponding to f. Thus $w = f(z)$ and $w = f^{-1}(z)$ are inverse functions of each other.

* LIMITS:- Let $w = f(z)$ be a single-valued function of z defined in some neighbourhood of a point $z = a$, except possibly at $z = a$ itself. We say that $f(z)$ tends to limit l as z tends to a along any path in a defined region, if for each arbitrary $\epsilon > 0$, $\exists$ a positive number δ depending upon ϵ such that

$$|f(z) - l| < \epsilon \quad \text{whenever} \quad 0 < |z - a| < \delta$$

Symbolically, $\lim_{z \to a} f(z) = l$.

NOTE: Geometrically, if 'a' is a point in the complex plane, then $\lim_{z \to a} f(z) = l$ if the difference in absolute value between $f(z)$ and l can be made as small as we wish by choosing points z sufficiently close to a (excluding $z = a$ itself).

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* Theorems on Limits:

If $\lim_{z \to z_0} f(z) = l_1$ and $\lim_{z \to z_0} g(z) = l_2$ then

i) $\lim_{z \to z_0} \{ f(z) \pm g(z) \} = \lim_{z \to z_0} f(z) \pm \lim_{z \to z_0} g(z) = l_1 \pm l_2$

ii) $\lim_{z \to z_0} \{ f(z) \cdot g(z) \} = \{ \lim_{z \to z_0} f(z) \} \{ \lim_{z \to z_0} g(z) \} = l_1 l_2$

iii) $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)} = \frac{l_1}{l_2}$ if $l_2 \neq 0$.

* Infinity:

By means of the transformation $w = \frac{1}{z}$ the point $z = 0$ (i.e origin) is mapped into $w = \infty$, called the point at infinity in the w-plane. Similarly we denote by $z = \infty$ the point at infinity in the z-plane. To consider the behaviour of $f(z)$ at $z = \infty$, it suffices to let $z = \frac{1}{w}$ and examine the behaviour of $f(1/w)$ at $w = 0$.

We say that $\lim_{z \to \infty} f(z) = l$ or $f(z)$ approaches l as z approaches infinity, if for any $\epsilon > 0$, $\exists$ $M > 0$ such that

$$|f(z) - l| < \epsilon \quad \text{whenever} \quad |z| > M.$$

We say that $\lim_{z \to z_0} f(z) = \infty$ or $f(z)$ approaches infinity as z approaches z_0, if for any $N > 0$ we can find $\delta > 0$ such that $|f(z)| > N$ whenever $0 < |z - z_0| < \delta$.

* Continuity:

A function $w = f(z)$ of complex variable z defined in a certain region D is said to be continuous at the point $z = z_0$ of D if for any $\epsilon > 0$, $\exists$ $\delta > 0$, depending on ϵ and z_0 such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{for all points } z \text{ of } D \text{ in the nbd } |z - z_0| < \delta.$$

Symbolically, $\lim_{z \to z_0} f(z) = f(z_0)$.

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* Uniformly Continuity:

A function $w = f(z)$ is said to be uniformly continuous in the domain D if for a given $\epsilon > 0$, we can find a number $\delta > 0$ depending upon ϵ only (not the pt. z_0) such that

$$|f(z_1) - f(z_2)| < \epsilon \quad \text{whenever } |z_1 - z_2| < \delta$$

where z_1, z_2 are any two points of the region D.

NOTE: i) A function $f(z)$ is said to be continuous in a region if it is continuous at all points of the region.

ii) If may be observed that continuity at every point of an open domain D does not neccessarily imply uniform continuity in D. If, however, the function $f(z)$ is continuous at every point of a closed and bounded domain (i.e Compact domain) D, then it is uniformly continuous in D.

* Discontinuity: Points in z plane where $f(z)$ fails to be continuous are called discontinuities of $f(z)$ and $f(z)$ is said to be discontinuous at these points.

If $\lim_{z \to z_0} f(z)$ exists but is not equal to $f(z_0)$, we call z_0 a removable discontinuity, since by $f(z_0)$ to be the same as $\lim_{z \to z_0} f(z)$ the function becomes continuous.

* NOTE: If $f(z)$ is continuous at $z = z_0$, we can write the limit in the suggestive form $\lim_{z \to z_0} f(z) = f\left(\lim_{z \to z_0} z\right)$.

* Differentiability:

If $f(z)$ is a single-valued function in some region R of the z-plane, the derivative of $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \qquad \cdots (1)$$

, provided that this limit exists independent of the manner in which $\Delta z \to 0$. In such case we say that $f(z)$ is

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differentiable at z.

NOTE: i) The above definition can be written in the form:

Given any $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon$$, provided that $0 < |z - z_0| < \delta$, along whatever path z approaches z_0 in the domain D.

ii) In order that a function $f(z)$ may be differentiable at any point z_0, is not enough that $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists; what is required further is that this limit must be unique. i.e it must have the same value along what ever the path z approaches z_0.

iii) Differentiability implies continuity but a continuous function is not necessarily differentiable.

* Ex: If $f(z) = z^2$, prove that $\lim_{z \to z_0} f(z) = z_0^2$

Soln: We must show that for each $\epsilon > 0$, $\exists\ \delta$ (depending on in general ϵ) s.t

$$|z^2 - z_0^2| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

Now, $|z^2 - z_0^2| = |(z - z_0)(z + z_0)| = |z - z_0||z + z_0|$

$$= |z - z_0||z - z_0 + 2z_0|$$

$$\leq |z - z_0|\{|z - z_0| + |2z_0|\}.$$

If $\delta \leq 1$ then $0 < |z - z_0| < \delta$ implies that

$$|z^2 - z_0^2| < \delta(1 + |2z_0|)$$

If we choose $\delta = \frac{\epsilon}{1 + |2z_0|} > 0$, then we have

$$|z^2 - z_0^2| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

Hence, $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} z^2 = z_0^2$

Hence the required result proved.

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* Ex: Prove that $\lim_{z\to i} \frac{3z^4-2z^3+8z^2-2z+5}{z-i} = 4+4i$

Soln: We must show that for any $\epsilon>0$ we can find $\delta>0$ such that $\left|\frac{3z^4-2z^3+8z^2-2z+5}{z-i} - (4+4i)\right| < \epsilon$ whenever $0<|z-i|<\delta$.

Since, $z \neq i$ we can write

$$\frac{3z^4-2z^3+8z^2-2z+5}{z-i} = \frac{[3z^3-(2-3i)z^2+(5-2i)z+5i](z-i)}{z-i}$$

$$= 3z^3-(2-3i)z^2+(5-2i)z+5i$$

Then we must show that for any $\epsilon>0$, we can find $\delta>0$ such that

$$|3z^3-(2-3i)z^2+(5-2i)z-4+i| < \epsilon \text{ whenever } 0<|z-i|<\delta.$$

If $\delta \leq 1$ then $0<|z-i|<\delta$ implies

$$\begin{aligned}
|3z^3-(2-3i)z^2+(5-2i)z-4+i| &= |z-i|\,|3z^2+(6i-2)z-1-4i| \\
&= |z-i|\,|3(z-i+i)^2+(6i-2)(z-i+i)-1-4i| \\
&= |z-i|\,|3(z-i)^2+(12i-2)(z-i)-10-6i| \\
&< \delta\{3|z-i|^2+|12i-2|\,|z-i|+|-10-6i|\} \\
&< \delta\,(3+13+12) \\
&= 28\delta
\end{aligned}$$

If we choose $\delta = \frac{\epsilon}{28} > 0$ then we have

$$\left|\frac{3z^4-2z^3+8z^2-2z+5}{z-i} - (4+4i)\right| < \epsilon \text{ whenever } 0<|z-i|<\delta.$$

i.e $\lim_{z\to i} \frac{3z^4-2z^3+8z^2-2z+5}{z-i} = 4+4i$

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*Ex: If $\lim_{z \to z_0} f(z)$ exists, prove that it must be unique.

Proof: If possible let $\lim_{z \to z_0} f(z) = l_1$ and $\lim_{z \to z_0} f(z) = l_2$

We are to show that $l_1 = l_2$.

Now, for any $\epsilon > 0$, $\exists\ \delta > 0$ such that

$|f(z) - l_1| < \epsilon/2$ whenever $0 < |z - z_0| < \delta$

and $|f(z) - l_2| < \epsilon/2$ whenever $0 < |z - z_0| < \delta$

Then $|l_1 - l_2| = |l_1 - f(z) + f(z) - l_2| \le |f(z) - l_1| + |f(z) - l_2|$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$

$= \epsilon$

Since, ϵ is arbitrary, however small. and so it must be zero. Thus, $l_1 = l_2$.

*Ex: Prove that $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exists.

Proof: If the limit is to exist it must be independ of the manner in which z approaches the point 0.

Let $z \to 0$ along x-axis. Then $y = 0$ and then

$z = x + iy = x$ and $\bar{z} = x - iy = x$

So, the required limit is $\lim_{z \to 0} \frac{\bar{z}}{z}$

$= \lim_{x \to 0} \frac{x}{x}$ [Since, $z = x + iy$ As $z \to 0$ along x-axis then $x \to 0$

$= \lim_{x \to 0} 1$

$= 1$

Again, let $z \to 0$ along y-axis. Then $x = 0$ and

$z = x + iy = iy$; $\bar{z} = x - iy = -iy$

So, the required limit is $\lim_{z \to 0} \frac{\bar{z}}{z}$

$= \lim_{y \to 0} \frac{-iy}{iy}$

$= \lim_{y \to 0} (-1)$

$= -1$.

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Since, the two approaches do not give the same result.
So, the limit does not exist.

* **Ex:** If $f(z) = \dfrac{x^2y(y-ix)}{x^4+y^2}$, $z \neq 0$

and $f(0) = 0$

Prove that $\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

Soln:

Now, $\lim\limits_{z\to 0} \dfrac{f(z)-f(0)}{z}$

$= \lim\limits_{\substack{x\to 0 \\ y\to 0}} \left[\dfrac{x^2y(y-ix)}{x^4+y^2} \cdot \dfrac{1}{x+iy}\right]$ [As $z = x+iy$ and $z \to 0$ $\Rightarrow x \to 0, y \to 0$.

Now, if $z \to 0$ along any radius vector $y = mx$ we have

$\lim\limits_{z\to 0} \dfrac{f(z)-f(0)}{z} = \lim\limits_{x\to 0} \left[\dfrac{mx^3(mx-ix)}{x^4+m^2x^2} \cdot \dfrac{1}{x+imx}\right]$

$= \lim\limits_{x\to 0} \left[\dfrac{-imx^3(x+imx)}{(x^4+m^2x^2)(x+imx)}\right]$

$= \lim\limits_{x\to 0} \left[\dfrac{-imx^3}{x^4+m^2x^2}\right]$

$= \lim\limits_{x\to 0} \left[\dfrac{-imx}{x^2+m^2}\right]$

$= 0.$

If we approaches $z \to 0$ along any curve $y = x^2$, then

$\lim\limits_{z\to 0} \dfrac{f(z)-f(0)}{z} = \lim\limits_{\substack{x\to 0 \\ y\to 0}} \left[\dfrac{x^2y(y-ix)}{(x^4+y^2)(x+iy)}\right]$

$= \lim\limits_{x\to 0} \left[\dfrac{x^4(x^2-ix)}{(x^4+x^4)(x+ix^2)}\right]$

$= \lim\limits_{x\to 0} \left[-\dfrac{ix^4(x+ix^2)}{(x^4+x^4)(x+ix^2)}\right]$

$= \lim\limits_{x\to 0} \left(-\dfrac{i}{2}\right)$

$= -\dfrac{i}{2}$

Hence, this shows that $\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ only any radius vector.

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* Analytic Functions:

If the derivative exists at all points z of a region D, then $f(z)$ is said to be analytic function in D.

The terms regular and holomorphic are sometimes used as synonyms for analytic.

A function $f(z)$ is said to be analytic at a point z_0 if there exists a neighbourhood $|z-z_0|<\delta$ at all points of which $f'(z)$ exists.

* **Ex:** Show that the function $f(z)=z^n$ where n is a positive integer is an analytic function.

Soln: By definition of derivative of a function, we have

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z}$$

Now, $f'(z)$ exists, provided the above limit exist as $\Delta z \to 0$ in any manner. Again, by binomial theorem which is purely algebraic identity and so holds for complex numbers as well, we have

$$f'(z) = \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[z^n + {}^nC_1 z^{n-1}(\Delta z) + {}^nC_2 z^{n-2}(\Delta z)^2 + \cdots \right.$$
$$\left. \cdots + (\Delta z)^n - z^n \right]$$

$$= \lim_{\Delta z \to 0} \left[{}^nC_1 z^{n-1} + {}^nC_2 z^{n-2}(\Delta z) + \cdots + (\Delta z)^{n-1} \right]$$

$$= {}^nC_1 z^{n-1}$$

$$= n z^{n-1}$$

Hence $f(z)$ is regular in every finite region and its derivative is nz^{n-1}.

Thus, $f(z)$ is an analytic function.

* Cauchy-Riemann Equations:

A necessary condition that $w = f(z) = u(x,y) + iv(x,y)$

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be analytic in a region D is that, in D, u and v satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\text{i.e } u_x = v_y \quad , \quad u_y = -v_x \qquad \longrightarrow (1)$$

If the partial derivatives in (1) are continuous in D, then the Cauchy-Riemann equations are sufficient conditions that $f(z)$ be analytic in D.

NOTE: The functions $u(x,y)$ and $v(x,y)$ are sometimes called Conjugate function. Given one we can find the other & so that $u + iv = f(z)$ is analytic.

* The necessary condition for $f(z)$ to be analytic:

Theorem: The necessary condition for the function $w = f(z) = u(x,y) + iv(x,y)$ to be analytic (i.e differentiable) at any point $z (= x+iy)$ of its region D is that four partial derivatives u_x, u_y, v_x, v_y should exist and satisfy Cauchy-Riemann equation $u_x = v_y$ and $u_y = -v_x$.

Proof: Suppose $f(z)$ is analytic in a region D at any point z. Since, $f(z)$ is analytic at a point z in the region D. Therefore,

$$\lim_{\delta z \to 0} \frac{\delta w}{\delta z} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \qquad \text{exist and unique.} \longrightarrow (1)$$

i.e independent of the path along which $\delta z \to 0$.

Let us take, $\delta z \to 0$ in the direction parallel to x-axis

then $\delta z = \delta x + i\delta y$

i.e $\delta z = \delta x$ [$\because \delta z \to 0$, parallel to x-axis]

As $\delta z \to 0$, $\delta x \to 0$ when δz is parallel to x-axis

Now, $f(z) = u(x,y) + iv(x,y)$

$\therefore f(z+\delta z) = u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)$

$= u(x+\delta x, y) + iv(x+\delta x, y)$ [$\because \delta y = 0$]

$\therefore \delta w = f(z+\delta z) - f(z)$

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$\therefore \Delta w = u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x,y) - iv(x,y)$

$= \{u(x+\Delta x, y) - u(x,y)\} + i\{v(x+\Delta x, y) - v(x,y)\}$

$\therefore \frac{\Delta w}{\Delta z} = \frac{\{u(x+\Delta x, y) - u(x,y)\} + i\{v(x+\Delta x, y) - v(x,y)\}}{\Delta z}$

$= \frac{\{u(x+\Delta x, y) - u(x,y)\} + i\{v(x+\Delta x, y) - v(x,y)\}}{\Delta x}$ $[\because \Delta z = \Delta x]$

$= \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\,\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$

$\therefore$ using this from (1) we get

$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta x \to 0} \left\{ \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\,\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right\}$

$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$

$\therefore f'(z) = u_x + i v_x \longrightarrow (2)$

Now since $f'(z)$ exists, the partial derivatives u_x & v_x should exist.

Similarly, let, $\Delta z \to 0$ in the direction parallel to y-axis

$\therefore \Delta z = \Delta x + i\Delta y$

So, we have $\Delta z = i\Delta y$ $(\because \Delta z \to 0$, parallel to y-axis$)$
$\therefore \Delta x \to 0.$

Now, $f(z) = u(x,y) + iv(x,y)$

$\therefore f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$

$= u(x, y+\Delta y) + iv(x, y+\Delta y)$ $[\because \Delta x \to 0]$

$\therefore \Delta w = f(z+\Delta z) - f(z)$

$= u(x, y+\Delta y) - u(x,y) + i\{v(x, y+\Delta y) - v(x,y)\}$

$\therefore \frac{\Delta w}{\Delta z} = \frac{\{v(x, y+\Delta y) - v(x,y)\} + i\{u(x, y+\Delta y) - u(x,y)\}}{i\,\Delta y}$

$= \frac{1}{i}\,\frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$

Using this from (1) we get

$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta y \to 0} \left\{ \frac{1}{i}\,\frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y} \right\}$

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$\therefore f'(z) = -iu_y + v_y \longrightarrow (3)$

Since, $f'(z)$ exists, then the partial derivatives u_y, v_y exists.

Finally, the uniqueness of limit of led to conclude that the limits in (2) and (3) should be identical, i.e

$$u_x + iv_x = -iu_y + v_y$$

Equating real and imaginary part, we get

$$u_x = v_y \text{ and } u_y = -v_x.$$

These two relations are called the Cauchy-Riemann equations.

Hence the theorem.

* Sufficient condition for $f(z)$ to be analytic:

Theorem: The one-valued function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D if the four partial derivatives u_x, v_x, u_y, v_y exist, are continuous and satisfy the Cauchy-Riemann equations at each point D.

Proof: We have $u = u(x,y)$ and $u + \Delta u = u(x+\Delta x, y+\Delta y)$

i.e $\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$

Now, $\Delta u = u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y) + u(x, y+\Delta y) - u(x,y) \longrightarrow (1)$

$= \Delta x \cdot u_x(x+\theta\Delta x, y+\Delta y) + \Delta y \cdot u_y(x, y+\theta'\Delta y)$

$0<\theta<1 \; ; \; 0<\theta'<1 \longrightarrow (2)$

[Since, by Mean Value theorem we know that if
i) $f(x)$ is continuous in $[a,b]$ ii) $f'(x)$ exist in (a,b)
then $f(a+h) - f(a) = h f'(a+\theta h) \; ; \; 0<\theta<1$]

Where u_x and u_y are given to be continuous at each point to the given region D.

Then, $|u_x(x+\theta\Delta x, y+\Delta y) - u_x(x,y)| < \epsilon$

and $|u_y(x, y+\theta'\Delta y) - u_y(x,y)| < \epsilon.$

Provided $|\Delta x| < \delta \; , |\Delta y| < \delta$

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Let, $u_x(x+\theta\Delta x, y+\Delta y) - u_x(x,y) = \alpha$

and $u_y(x+\theta'\Delta x, y+\theta'\Delta y) - u_y(x,y) = \beta$

where $|\alpha| < \epsilon$ and $|\beta| < \epsilon$.

Then using the above relation we have from (2)

$$\Delta u = [u_x(x,y) + \alpha]\Delta x + [u_y(x,y) + \beta]\Delta y \longrightarrow (3)$$

Similarly, we shall get

$$\Delta v = [v_x(x,y) + \alpha']\Delta x + [v_y(x,y) + \beta']\Delta y \longrightarrow (4)$$

$$|\alpha'| < \epsilon' \text{ and } |\beta'| < \epsilon'$$

Hence, we get

$$\Delta w = \Delta u + i\Delta v$$

$$= [(u_x + \alpha)\Delta x + (u_y + \beta)\Delta y] + i[(v_x + \alpha')\Delta x + (v_y + \beta')\Delta y]$$

$$= [(u_x + iv_x)\Delta x + (u_y + iv_y)\Delta y + (\alpha + i\alpha')\Delta x + (\beta + i\beta')\Delta y]$$

$$= [(u_x + iv_x)\Delta x + i(iv_x + u_x)\Delta y + (\alpha + i\alpha')\Delta x + (\beta + i\beta')\Delta y]$$

$$[\because u_x = v_y \;;\; u_y = -v_x]$$

$$= (u_x + iv_x)(\Delta x + i\Delta y) + (\alpha + i\alpha')\Delta x + (\beta + i\beta')\Delta y$$

Dividing throughout by $\Delta z = \Delta x + i\Delta y$ we get

$$\frac{\Delta w}{\Delta z} = (u_x + iv_x) + \frac{(\alpha + i\alpha')\Delta x}{\Delta x + i\Delta y} + \frac{(\beta + i\beta')\Delta y}{\Delta x + i\Delta y}$$

or, $$\left|\frac{\Delta w}{\Delta z} - (u_x + iv_x)\right| = \left|\frac{(\alpha + i\alpha')\Delta x}{\Delta x + i\Delta y} + \frac{(\beta + i\beta')\Delta y}{\Delta x + i\Delta y}\right|$$

$$\leq \frac{|\alpha + i\alpha'||\Delta x|}{|\Delta x + i\Delta y|} + \frac{|\beta + i\beta'||\Delta y|}{|\Delta x + i\Delta y|}$$

$$\leq |\alpha + i\alpha'| + |\beta + i\beta'|$$

$$[\because |\Delta x| \leq |\Delta x + i\Delta y| \;;\; \because |\Delta y| \leq |\Delta x + i\Delta y|]$$

$$\leq |\alpha| + |\alpha'| + |\beta| + |\beta'|$$

$$\therefore \left|\frac{\Delta w}{\Delta z} - (u_x + iv_x)\right| < 2\epsilon + 2\epsilon'$$

Hence $f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = u_x + iv_x \longrightarrow (5)$

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Since, u_x, v_x exist and are unique.

Therefore, from expression (5) we conclude that $f'(z)$ exists.

Thus $f(z)$ is analytic at an arbitrary point z of D and hence it is analytic in the domain D.

Hence the theorem.

* **Remark:** The sufficient condition of an analytic function $f(z)$ is not true always.

for example we consider that $f(z) = \dfrac{xy^2(x+iy)}{x^2+y^4}$, $z \neq 0$

$= 0$, $z = 0$.

Now, $f(z) = \dfrac{xy^2(x+iy)}{x^2+y^4}$

$= \dfrac{x^2y^2}{x^2+y^4} + i\,\dfrac{xy^3}{x^2+y^4}$

$= u(x,y) + i\,v(x,y)$

$\therefore u(x,y) = \dfrac{x^2y^2}{x^2+y^4}$ and $v(x,y) = \dfrac{xy^3}{x^2+y^4}$.

Now, $u_x = \dfrac{\partial u}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{u(\Delta x + \Delta x, y) - u(x,y)}{\Delta x}$

At $z = 0$ i.e $x + iy = 0 \Rightarrow x = 0, y = 0$.

$\therefore u_x(0,0) = \lim\limits_{\Delta x \to 0} \dfrac{u(\Delta x, 0) - u(0,0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{0-0}{\Delta x} = 0$

$\therefore u_y(0,0) = \lim\limits_{\Delta y \to 0} \dfrac{u(0, \Delta y) - u(0,0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{0-0}{\Delta y} = 0$

$\therefore v_x(0,0) = \lim\limits_{\Delta x \to 0} \dfrac{v(\Delta x, 0) - v(0,0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{0-0}{\Delta x} = 0$

$\therefore v_y(0,0) = \lim\limits_{\Delta y \to 0} \dfrac{v(0, \Delta y) - v(0,0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{0-0}{\Delta y} = 0$

Hence, at the origin $(0,0)$ i.e $z = 0$

$u_x = v_y$, $u_y = -v_x$

$\therefore$ It Satisfy the Cauchy-Riemann equation obviously.

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But $f(z)$ is not derivable at $z=0$. In fact, it fails to have a unique derivative at $z=0$.

Now, $f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$

$= \lim_{\substack{x\to 0 \\ y\to 0}} \frac{xy^2(x+iy)}{(x^2+y^4)(x+iy)}$ [Since $z = x+iy$, $z\to 0 \Rightarrow x\to 0, y\to 0$]

$= \lim_{\substack{x\to 0 \\ y\to 0}} \frac{xy^2}{x^2+y^4}$

We approach to the origin i.e $z\to 0$ along the curve $x = my^2$ where m is arbitrary and as $x\to 0$, $y\to 0$

$\therefore f'(0) = \lim_{y\to 0} \frac{my^4}{(m^2+1)y^4}$

$= \lim_{y\to 0} \frac{m}{1+m^2}$

$= \frac{m}{1+m^2}$

which is different for different values of m.

$\therefore f'(0)$ does not exist.

Hence, the sufficient condition is not true always.

* **Conjugate Function:**

If a function $f(z) = u(x,y) + iv(x,y)$ analytic in a domain D, then the functions u and v of two variables x, y are called conjugate functions.

* **Laplace's Equation:**

Theorem: Real and imaginary parts of an analytic function satisfy Laplace's equation.

Proof: Let the function $f(z) = u+iv$ be analytic in some region D. Then by Cauchy-Riemann equations, we have

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ⟶ (1) and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ⟶ (2)

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Again diff. (1) & (2) w.r.to x & y respt. we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) \quad [\text{using (2)}]$$

$$= -\frac{\partial^2 u}{\partial y^2}$$

Hence, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ i.e $u_{xx} + u_{yy} = 0 \longrightarrow (3)$

Similarly, $\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$

$$= -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) \quad [\text{using (1)}]$$

$$= -\frac{\partial^2 v}{\partial y^2}$$

So that, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ i.e $v_{xx} + v_{yy} = 0 \longrightarrow (4)$

The equation (3) & (4) shows that u & v satisfy Laplace's equation in two dimensions

i.e $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$.

* **Harmonic Functions:**

Definition: A real-valued function $u(x,y)$ is said to be harmonic in a domain D if for all points (x,y) in D, all second order partial derivatives exist and are continuous, and satisfy Laplace's equation.

i.e $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Thus, if $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then $u(x,y)$ and $v(x,y)$ are harmonic in D. In such a case, u and v are called conjugate harmonic functions or simply conjugate functions.

* **NOTE:** If harmonic functions u and v satisfy Cauchy-Riemann equations, then $u+iv$ is an analytic function.

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*Ex: Show that the function $e^x(\cos y + i \sin y)$ is holomorphic and find its derivative.

Proof:

Let $f(z) = u + iv = e^x(\cos y + i \sin y)$

$\therefore u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$.

If the function $f(z)$ is holomorphic (i.e analytic), then it must satisfy Cauchy-Riemann equation.

Now, $\frac{\partial u}{\partial x} = e^x \cos y$; $\frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y$; $\frac{\partial v}{\partial y} = e^x \cos y$

From these we have, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

i.e the Cauchy-Riemann cond^{n}s are satisfied.

Therefore, the given function is holomorphic.

2nd part:

Now, $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

$= e^x \cos y + i e^x \sin y$

$= e^x(\cos y + i \sin y)$

$= e^x \cdot e^{iy}$

$= e^{(x+iy)}$

$= e^z$ [$\because z = x + iy$]

*Ex: Prove that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic and find its harmonic conjugate.

Soln^n: If u is harmonic function, then it satisfy Laplace eqn^n $\nabla^2 u = 0$. i.e $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Now, $\frac{\partial u}{\partial x} = e^{-x}(\sin y) - e^{-x}(x \sin y - y \cos y)$

$\therefore \frac{\partial^2 u}{\partial x^2} = -e^{-x} \sin y + e^{-x}(x \sin y - y \cos y) - e^{-x} \sin y$

$= -2e^{-x} \sin y + e^{-x}(x \sin y - y \cos y)$

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and $\frac{\partial u}{\partial y} = e^{-x}(x\cos y - \cos y + y\sin y)$

$\therefore \frac{\partial^2 u}{\partial y^2} = e^{-x}(-x\sin y + \sin y + \sin y + y\cos y)$

$= 2\sin y\, e^{-x} + e^{-x}(-x\sin y + y\cos y)$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2e^{-x}\sin y + e^{-x}(x\sin y - y\cos y) + 2\sin y\, e^{-x}$

$+ e^{-x}(-x\sin y + y\cos y)$

$= 0.$

$\therefore u$ is a harmonic function.

2nd part: Let v be its conjugate harmonic function, we have

$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \left[\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;;\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right].$

$= -e^{-x}(x\cos y - \cos y + y\sin y)dx + e^{-x}(\sin y - x\sin y + y\cos y)dy$

$= M\,dx + N\,dy \longrightarrow (1)$

where $M = -e^{-x}(x\cos y - \cos y + y\sin y)$

$\therefore \frac{\partial M}{\partial y} = -e^{-x}(-x\sin y + 2\sin y + y\cos y)$

and $N = e^{-x}(\sin y - x\sin y + y\cos y)$

$\therefore \frac{\partial N}{\partial x} = -e^{-x}(\sin y - x\sin y + y\cos y) - e^{-x}\sin y)$

$= -e^{-x}(-x\sin y + 2\sin y + y\cos y)$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence (1) is an exact differential eqn^n.

Now, $\int M\,dx = \int -e^{-x}(x\cos y - \cos y + y\sin y)dx$

$= -\cos y \int x e^{-x}dx + \cos y \int e^{-x}dx - y\sin y \int e^{-x}dx$

$= -\cos y\,(-xe^{-x} - e^{-x}) - e^{-x}\cos y + y\sin y\, e^{-x}$, keeping y is fixed

$+ C$ and C is independent of x

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$= e^{-x}(y\sin y + x\cos y) + c_1$

and $\int N\,dy = \int e^{-x}(\sin y - x\sin y + y\cos y)\,dy$

$= e^{-x}\left(\int \sin y\,dy - x\int \sin y\,dy + \int y\cos y\,dy\right)$

$= e^{-x}(-\cos y + x\cos y + y\sin y + \cos y) + c_2$, keeping x is const.

$= e^{-x}(y\sin y + x\cos y) + c_2$ and c_2 is independent of y

Hence,

$v = e^{-x}(y\sin y + x\cos y) + c$, c = constant.

* **Ex:** For what values of z do the function w defined by the following equations ceases to be analytic?

$z = \sin u\cosh v + i\cos u\sinh v$; $w = u + iv$.

Soln:

NOTE: The function $w = f(z)$ i.e $z = f^{-1}(w) = F(w)$ will ceases to be analytic where $f'(z) = 0$ or say where $\frac{dw}{dz} = 0$.

If immediately follows that the function $w = f(z)$ where $z = F(w)$ will cease to be analytic where $\frac{dz}{dw} = 0$.

If may be seen that when $z = x + iy$ and $w = f(z)$ we have $\frac{dw}{dz} = \frac{\partial w}{\partial x}$

Therefore if $w = u + iv$ and $z = F(w)$ we should have

$$\frac{dz}{dw} = \frac{\partial z}{\partial u}.$$

Soln: Here $z = \sin u\cosh v + i\cos u\sinh v$ ----(1)

$\therefore \frac{\partial z}{\partial u} = \cos u\cosh v - i\sin u\sinh v$

We have $\frac{dz}{dw} = \frac{\partial z}{\partial u} = \cos u\cosh v - i\sin u\sinh v$. ⟶(2)

Squaring and adding (1) & (2) we have

$z^2 + \left(\frac{dz}{dw}\right)^2 = \sin^2 u\cosh^2 v - \cos^2 u\sinh^2 v + \cos^2 u\cosh^2 v - \sin^2 u\sinh^2 v$

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$$\text{i.e } \left(\frac{dz}{dw}\right)^2 + z^2 = \cosh^2 v(\cos^2 u + \sin^2 u) - \sinh^2 v(\cos^2 u + \sin^2 u)$$

$$= \cosh^2 v - \sinh^2 v = 1$$

$$\text{i.e } \frac{dz}{dw} = \sqrt{1 - z^2}$$

Hence, w ceases to be analytic if $\frac{dz}{dw} = 0$

$$\text{i.e } \sqrt{1-z^2} = 0$$

$$\Rightarrow z = \pm 1$$

$\therefore$ for $z = \pm 1$, the given function w ceases to be analytic.

***Ex:** For what values of z the function w defined by the following equations ceases to be analytic?

$$z = e^{-v}(\cos u + i \sin u), \quad w = u + iv.$$

Soln: Here, $z = e^{-v}(\cos u + i \sin u)$

$$\therefore \frac{\partial z}{\partial u} = e^{-v}(-\sin u + i \cos u) = -i\, e^{-v}(\cos u + i \sin u)$$

$$= -iz$$

We have, $\frac{dz}{dw} = \frac{\partial z}{\partial u}$

$$\therefore \frac{dz}{dw} = -iz$$

Hence, w ceases to be analytic when $\frac{dz}{dw} = 0$

$$\text{i.e, } iz = 0$$

$$\text{i.e } z = 0.$$

*** Ex:** For what values of z the function defined by the following equation ceases to be analytic

Try yourself. Ans $\Rightarrow z = \pm i$

*** Ex:** Show that the function $f(z) = \sin x \cosh y + i \cos x \sinh y$ is continuous as well as analytic every where.

Soln: Let, $f(z) = u(x,y) + i\, v(x,y)$

then $u(x,y) = \sin x \cosh y$, $v(x,y) = \cos x \sinh y$

Since, u and v both are rational functions of x and y, whose denominators are non-zero for all values of

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x and y, therefore u and v are both continuous everywhere.

Hence, $f(z)$ is also continuous everywhere.

Again, $\frac{\partial u}{\partial x} = \cos x \cosh y$; $\frac{\partial u}{\partial y} = \sin x \sinh y$

$\frac{\partial v}{\partial x} = -\sin x \sinh y$; $\frac{\partial v}{\partial y} = \cos x \cosh y$

The four partial derivatives are rational functions of x and y with non-zero denominators for all values of x and y, therefore they are continuous everywhere.

Also, here $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

i.e Cauchy-Riemann conditions are satisfied.

Thus, the four partial derivatives being continuous everywhere and Cauchy-Riemann equations being satisfied, the function $f(z)$ is analytic everywhere.

* Ex: Show that the function $f(z) = xy + iy$ is everywhere continuous but is not analytic

Soln: Let, $f(z) = u + iv$ where $u = xy$, $v = y$.

We see that both u and v being polynomials in x and y are continuous everywhere. Hence $f(z)$ is continuous everywhere.

Here $\frac{\partial u}{\partial x} = y$, $\frac{\partial u}{\partial y} = x$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 1$

Thus, we see that $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

i.e, Cauchy-Riemann conditions are not satisfied anywhere. Hence $f(z)$ is not analytic at any point.

* Ex: Show that the functions $f(z) = e^{-z^{-4}}$; $z \neq 0$

$= 0$; $z = 0$

is not analytic at $z = 0$, although Cauchy-Riemann Equations are satisfied at the point. How would you

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explain this.

Sol^{n}: Here $u+iv = f(z) = e^{-z^{-4}} = e^{-(x+iy)^{-4}}$

$$= e^{-\frac{1}{(x+iy)^4}}$$

$$= e^{-\frac{(x-iy)^4}{(x^2+y^2)^4}}$$

or, $u+iv = e^{-\frac{1}{(x^2+y^2)^4}} \{(x^4+y^4-6x^2y^2) - i\,4xy(x^2-y^2)\}$

$$\therefore u+iv = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}} \left\{\cos\left(\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right) + i\sin\left(\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right)\right\}$$

$$\therefore u = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}} \cos\left(\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right)$$

and $$v = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}} \sin\left(\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right)$$

At $z=0$, $\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{e^{-x^{-4}}}{x} = 0$

$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{-e^{-y^{-4}}}{y} = 0$

$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{0}{x} = 0$

$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{0}{y} = 0$

Hence Cauchy-Riemann conditions are satisfied at $z=0$.

But, $f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{e^{-z^{-4}}}{z}$

$$= \lim_{r\to 0} \frac{e^{-r^{-4}}}{r e^{i\pi/4}} \quad \text{if } z\to 0 \text{ along } z = re^{i\pi/4}.$$

$$= \infty$$

Showing that $f'(z)$ does not exist at $z=0$.

Hence, $f(z)$ may not be analytic at $z=0$.

* NOTE: $f(z)$ may not be analytic at $z=z_0$ if merely the Cauchy-Riemann conditions are satisfied there. In addition

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to this $f'(z)$ must exist and should be unique at z and at every point in the neighbourhood of z_0.

* **Ex:** Prove that the function $f(z) = u + iv$ where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} \quad ; z \neq 0$$

$$= 0 \quad ; z = 0$$

is continuous and that Cauchy-Riemann equations are satisfied at the origin, yet $f'(z)$ does not exist there.

Soln: Here, Let, $f(z) = u(x,y) + iv(x,y)$

where, $u(x,y) = \frac{x^3 - y^3}{x^2+y^2}$; $v(x,y) = \frac{x^3+y^3}{x^2+y^2}$ where $z \neq 0$.

Here we see that both $u(x,y)$ and $v(x,y)$ are rational and finite for all values of $z \neq 0$, So u and v are continuous at all those points for which $z \neq 0$. Hence $f(z)$ continuous where $z \neq 0$.

At the origin, $u = 0$, $v = 0$ [$\because f(0) = 0$]

Hence u & v are both continuous at the origin.

Consequently $f(z)$ is continuous everywhere.

At the origin,

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \left(\frac{x}{x}\right) = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \left(-\frac{y}{y}\right) = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x \to 0} \left(\frac{x}{x}\right) = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y \to 0} \left(\frac{y}{y}\right) = 1$$

We observe that, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence Cauchy-Riemann conditions are satisfied at $z = 0$

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2nd part:

Now, $f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$

$= \lim_{\substack{x\to 0 \\ y\to 0}} \frac{x^3(1+i)-y^3(1-i)}{(x^2+y^2)(x+iy)}$ { Since, $z = x+iy$, $z\to 0 \Rightarrow x\to 0, y\to 0$

$= \lim_{\substack{x\to 0 \\ y\to 0}} \frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$

Let, $z\to 0$ along the radius vector $y = mx$ where m is arbitrary and as $y\to 0$, $x\to 0$.

$\therefore f'(0) = \lim_{x\to 0} \frac{(x^3-m^3x^3)+i(x^3+m^3x^3)}{(x^2+m^2x^2)(x+imx)}$

$= \lim_{x\to 0} \frac{x^3\{(1-m^3)+i(1+m^3)\}}{x^3\{(1+m^2)(1+im)\}}$

$= \frac{1-m^3+i(1+m^3)}{(1+m^2)(1+im)}$

which is different for different realues of m.

i.e $f'(0)$ is not unique and hence $f'(0)$ does not exist.

Thus, $f'(z)$ does not exist at the origin.

* Ex: Examine the nature of the function

$$f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}} \quad ; z\neq 0$$

$$f(0) = 0$$

in the origin including the origin.

Hint → Same as above.

* Ex: Show that the function $f(z) = \sqrt{|xy|}$ is not regular at the origin, although Cauchy-Riemann equations are satisfied at the point.

Solⁿ: Let the function be $f(z) = u(x,y) + iv(x,y)$

Here, $u(x,y) = \sqrt{|xy|}$ and $v(x,y) = 0$

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At the origin.

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

Hence, Cauchy - Riemann equations are satisfied at the origin.

Again, $f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{\substack{x\to 0 \\ y\to 0}} \frac{\sqrt{|xy|}}{x+iy}$

Taking $z\to 0$ along $y = mx$ and $y\to 0$, $x\to 0$

$$\therefore f'(0) = \lim_{x\to 0} \frac{\sqrt{|mx^2|}}{x+imx}$$

$$= \lim_{x\to 0} \frac{\sqrt{m}}{1+im}$$

$$= \frac{\sqrt{m}}{1+im}.$$

which is different for different m.

i.e $f'(0)$ is not unique.

Hence, $f(z)$ is not analytic at the origin although Cauchy - Riemann equations are satisfied.

* **Orthogonal System:**

Two curves are said to be orthogonal to each other when they intersect at right angle at each of their points of intersection.

To show that, if $w = f(z) = u+iv$ be analytic function of $z = x+iy$, then the family of curves $u(x,y) = c_1$ and $v(x,y) = c_2$ form an orthogonal system

For the curve $u(x,y) = c_1$, we have on differentiating

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \quad \text{i.e} \quad \frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} \longrightarrow (1)$$

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And for the curve $v(x,y) = c_2$, we have on differentiating

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0 \quad \text{i.e} \quad \frac{dy}{dx} = -\frac{\partial v/\partial x}{\partial v/\partial y} \longrightarrow (2)$$

Now the product of (1) & (2) is

$$\left(-\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}\right)\cdot\left(-\partial v/\partial x\Big/\partial v/\partial y\right)$$

$$= \left(\partial v/\partial y\Big/\partial v/\partial x\right)\left(-\frac{\partial v}{\partial x}\Big/\frac{\partial v}{\partial y}\right)$$

$$= -1$$

Hence the two curves $u(x,y) = c_1$ & $u(x,y) = c_2$ are orthogonal and since, c_1 & c_2 are parameters, therefore we say that the two family of curves $u(x,y) = c_1$ and $u(x,y) = c_2$ form an orthogonal system.

* **Theorem:** If $f(z) = u + iv$ is an analytic function, then u and v are both harmonic functions.

Proof: Let, $f(z) = u + iv$, be an analytic function, then we have

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \text{or} \quad \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\} \longrightarrow (1)$$

Cauchy-Riemann equations.

Also, because u and v are the real and imaginary parts of an analytic function, therefore derivatives of u and v, of all orders, ~~so that we have~~ exists and are continuous fun^c of x, y

So, we have $\dfrac{\partial^2 v}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial y \partial x} \longrightarrow (2)$

Differentiating eqn (1) we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

Adding these and using (2) we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly, $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

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Hence both u and v satisfy Laplace's eqn^n

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Therefore both u and v are harmonic functions.

*Ex: Show that the function $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic and find its harmonic conjugate.

Soln: We have $u = \frac{1}{2}\log(x^2+y^2)$

$$\therefore \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \quad \therefore \frac{\partial^2 u}{\partial x^2} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

and $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$, $\frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

So that, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0$

Hence, u is a harmonic function.

We have, $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ [by Cauchy-Riemann eqn^n $u_x = v_y$; $u_y = -v_x$

$= -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$

$= \frac{x\,dy - y\,dx}{x^2+y^2}$

$= d\left(\tan^{-1}(y/x)\right)$

Integrating it, we get

$\therefore v = \tan^{-1}(y/x) + C$ where C is real constant.

* Ex: Show that the function $u = \cos x \cosh y$ is harmonic and find its harmonic conjugate.

Hint: Same as above and $v = -\sin x \sinh y + c$

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*Ex: Prove that $u = y^3 - 3x^2y$ is a harmonic function. Determine its harmonic conjugate and find the corresponding analytic function $f(z)$ in terms of z.

Soln: We have $u = y^3 - 3x^2y$

$$\therefore -\frac{\partial u}{\partial x} = -6xy \; ; \; \frac{\partial u}{\partial y} = 3y^2 - 3x^2$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = -6y \; ; \; \frac{\partial^2 u}{\partial y^2} = 6y$$

So that u satisfies Laplace's equation ; hence u is a harmonic function.

Let v be the harmonic conjugate to u, we have

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$= -(3y^2 - 3x^2)dx - 6xy\,dy$$

$$= -(3y^2dx + 6xy\,dy) + 3x^2dx$$

$$= -d(3xy^2) + 3x^2dx$$

Integrating, $v = -3xy^2 + x^3 + C$

which is the harmonic conjugate to u.

$$f(z) = u + iv$$

$$= (y^3 - 3x^2y) + i(-3xy^2 + x^3 + C)$$

$$= y^3 - 3x^2y - 3ixy^2 + ix^3 + iC$$

$$= i(x^3 + 3ix^2y + 3xy^2 + (iy)^3) + iC$$

$$= i(x + iy)^3 + iC$$

$$= iz^3 + iC$$

*Ex: Prove that if $u = x^2 - y^2$, $v = -y/(x^2 + y^2)$ both u and v satisfy Laplace's equation, but $u + iv$ is not an analytic function of z.

Hint: $f(z) = u + iv$, Now $\frac{\partial u}{\partial x} = 2x$; $\frac{\partial u}{\partial y} = -2y$; $\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$;

$\frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(y^2+x^2)^2}$, Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\therefore u$ & v satisfy Laplace eqn.

but $u_x \neq v_y$, $u_y \neq -v_x$, so $u + iv$ is not analytic.

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is a function of z in any domain

* Polar form of Cauchy-Riemann Equations:

Theorem: If $z = re^{i\theta}$, $f(z) = u + iv$ where r, θ, u and v are real and $f(z)$ is analytic function. Show that Cauchy-Riemann differential equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad -\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r}$$

Proof: We have $z = re^{i\theta}$

$$\therefore x + iy = r(\cos\theta + i\sin\theta)$$

$$\therefore x = r\cos\theta, \quad y = r\sin\theta$$

$$\therefore x^2 + y^2 = r^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \quad \therefore \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Now,
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$
$$= \frac{\partial u}{\partial r}\frac{x}{r} + \frac{\partial u}{\partial \theta}\left(-\frac{y}{r^2}\right)$$
$$= \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \longrightarrow (1)$$

$\because \frac{\partial r}{\partial x} = \frac{x}{r}$

$\because \frac{\partial \theta}{\partial x} = \frac{1}{1+y^2/x^2}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{y}{r^2}$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$
$$= \frac{\partial u}{\partial r}\frac{y}{r} + \frac{\partial u}{\partial \theta}\left(\frac{x}{r^2}\right)$$
$$= \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} \longrightarrow (2)$$

$$\therefore \frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x}$$
$$= \frac{\partial v}{\partial r}\cdot\frac{x}{r} + \frac{\partial v}{\partial \theta}\left(-\frac{y}{r^2}\right)$$
$$= \cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta} \longrightarrow (3)$$

$$\therefore \frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y}$$
$$= \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta} \longrightarrow (4)$$

By Cauchy-Riemann equation we have

$$u_x = v_y$$

i.e,
$$\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta} \longrightarrow (5)$$

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and $u_y = -v_x$

or, $\sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r} \frac{\partial u}{\partial \theta} = \frac{\sin\theta}{r} \frac{\partial v}{\partial \theta} - \cos\theta \frac{\partial v}{\partial r}$ ....(6)

Multiplying (5) by $\cos\theta$ and (6) by $\sin\theta$ and adding we get

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

Multiplying (5) by $\sin\theta$ and (6) by $\cos\theta$ and subtracting we get $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$.

These are the Cauchy - Riemann equation in polar form.

* Ex: If n is real, show that $r^n(\cos n\theta + i \sin n\theta)$ is analytic except possibly when $r=0$, and that its derivative is

$$n r^{n-1} [\cos(n-1)\theta + i \sin(n-1)\theta].$$

Soln: Let $w = r^n(\cos n\theta + i \sin n\theta) \equiv f(z) \equiv u + iv$

So here $u = r^n \cos n\theta$; $v = r^n \sin n\theta$

then $\frac{\partial u}{\partial r} = n r^{n-1} \cos n\theta$; $\frac{\partial v}{\partial r} = n r^{n-1} \sin n\theta$

$\frac{\partial u}{\partial \theta} = -n r^n \sin n\theta$; $\frac{\partial v}{\partial \theta} = n r^n \cos n\theta$

Thus, we see that $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$

and $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$

That is, Cauchy - Riemann conditions are satisfied.

Hence the function $w = r^n(\cos n\theta + i \sin n\theta)$ is analytic, if $\frac{dw}{dz}$ exists for all finite values of z.

We have $\frac{dw}{dz} = (\cos\theta - i \sin\theta) \frac{\partial w}{\partial r}$

$= (\cos\theta - i \sin\theta)\, n r^{n-1} (\cos n\theta + i \sin n\theta)$

$= n r^{n-1} \{\cos\theta \cos n\theta + \sin\theta \sin n\theta\}$
$+ i (\cos\theta \sin n\theta - \sin\theta \cos n\theta)\}$

$= n r^{n-1} \{\cos(n-1)\theta + i \sin(n-1)\theta\}$

This exists for all finite values of r including zero, except when $r=0$ and $n \le 1$.

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* **Construction of a function $f(z)$ when one conjugate function is given:**

Let, $f(z) = u + iv$ be analytic function; where u & v are the conjugate functions.

When one of these two functions is given, the function $f(z)$ can be constructed in two different ways.

i) only v is findout and then construct $f(z)$

and ii) the function $f(z)$ as a whole is directly find out.

The second method to construct $f(z)$ directly without finding v is due to Milne Thomson and this method is given below:

Since, $z = x + iy \quad \therefore \bar{z} = x - iy$

$\therefore x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$

$\therefore f(z) = u(x, y) + i v(x, y)$

$= u\left\{\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right\} + i v\left\{\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right\}$

This relation can be regarded as a formal identity in two independent variable z and $\bar{z}$.

Putting $z = \bar{z}$ in above we get

$f(z) = u(z, 0) + i v(z, 0)$

Now, $f(z) = u + iv \quad \therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ by Cauchy-Riemann eqn.

Hence if we write $\frac{\partial u}{\partial x} \equiv \phi_1(x, y)$ and $\frac{\partial u}{\partial y} \equiv \phi_2(x, y)$

$\therefore f'(z) = \phi_1(x, y) - i\phi_2(x, y)$

$= \phi_1(z, 0) - i\phi_2(z, 0)$

Integrating if we write

$f(z) = \int [\phi_1(z, 0) - i\phi_2(z, 0)]\, dz + C$

where C is an arbitrary constant.

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Thus the function $f(z)$ is constructed when $u(x,y)$ is given.

Similarly, if $v(x,y)$ is given, it can be shown that

$$f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)]dz + C$$

where $\psi_1(x,y) = \frac{\partial v}{\partial y}$ and $\psi_2(x,y) = \frac{\partial v}{\partial x}$.

* **Ex:** Prove that the function $u = e^x(x\cos y - y\sin y)$ satisfies Laplace's equations and find the corresponding analytic function $f(z) = u + iv$.

Soln: Here, $u = e^x(x\cos y - y\sin y)$

$$\therefore \frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y$$
$$= \phi_1(x,y).$$

$$\therefore \frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y)$$
$$= \phi_2(x,y).$$

By Milne's method we have

$$f'(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]dz + C$$

$$= \int [e^z(z+1) - i\cdot 0]dz + C$$

$$= \int ze^z dz + \int e^z dz + C$$

$$= ze^z - e^z + e^z + C$$

$$\therefore f(z) = ze^z + C$$

* **Ex:** If $u = \frac{\sin 2x}{\cosh 2y + \cos 2x}$, find the corresponding analytic function $f(z) = u + iv$

Soln: We have, $u = \frac{\sin 2x}{\cosh 2y + \cos 2x}$

$$\therefore \frac{\partial u}{\partial x} = \frac{2\cos 2x(\cosh 2y + \cos 2x) + 2\sin^2 2x}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2 + 2\cos 2x\cosh 2y}{(\cosh 2y + \cos 2x)^2} = \phi_1(x,y) \text{ (say)}.$$

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and $\frac{\partial u}{\partial y} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2} = \phi_2(x,y)$ (Say)

$\therefore$ By Milne's method we have

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]dz + C$$

$$= \int \left[\frac{2+2\cos 2z}{(1+\cos 2z)^2} - i\cdot 0\right]dz + C$$

$$= 2\int \frac{dz}{1+\cos 2z} + C$$

$$= 2\int \frac{dz}{2\cos^2 z} + C$$

$$= \int \sec^2 z\, dz + C$$

$$\therefore f(z) = \tan z + C$$

*Ex: Prove that the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ satisfies Laplace's equation and determine the corresponding analytic function $u+iv$.

Hint: Show $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$; $f(z) = z^3 + 3z^2 + C$.

*Ex: If $f(z) = u+iv$ is an analytic function of $z = x+iy$ and $u - v = e^x(\cos y - \sin y)$, find $f(z)$ interms of z.

Soln: We have $u+iv = f(z)$

$\therefore iu - v = if(z)$

Adding these we have

$(1+i)u + (i-1)v = (1+i)f(z) = F(z) = U + iV$ (Say)

where $F(z)$ is analytic because $f(z)$ is so.

$\therefore (U-v) + i(u+v) = U + iV$

$\therefore u - v = U$ and $u + v = V$

Therefore, $U = u - v = e^x(\cos y - \sin y)$

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$\therefore \frac{\partial v}{\partial x} = e^x(\cos y - \sin y) = \phi_1(x,y)$

and $\frac{\partial v}{\partial y} = e^x(-\sin y - \cos y) = \phi_2(x,y)$

By Milne's method we get

$f'(z) = \phi_1(z,0) - i\,\phi_2(z,0)$

$= (e^z + i e^z)$

$= (1+i)\,e^z$

Integrating, we get $f(z) = (1+i)\int e^z dz + C$

$\therefore (1+i) f(z) = (1+i)\,e^z + C$

$\therefore f(z) = e^z + c_1$

*Ex: If $u - v = (x-y)(x^2+4xy+y^2)$ and $f(z) = u+iv$ is an analytic function of $z = x+iy$, find $f(z)$ in terms of z.

Soln: $u + iv = f(z)$

$\therefore iu - v = if(z)$

Adding these, $(u-v) + i(u+v) = (1+i)f(z) = F(z)$ (say)

where $F(z)$ is analytic because $f(z)$ is so.

Let, $U = u - v$, $u + v = V$, then $F(z) = U + iV$ is an analytic function.

We have, $U = u - v = (x-y)(x^2+4xy+y^2)$

$\therefore \frac{\partial U}{\partial x} = 3x^2 + 6xy - 3y^2 = \phi_1(x,y)$

and $\frac{\partial U}{\partial y} = 3x^2 - 6xy - 3y^2 = \phi_2(x,y)$

By Milne's method, we have

$F(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + C$

$= \int (3z^2 - i\,3z^2)\,dz + C$

i.e $(1+i) f(z) = (1-i) z^3 + C$

$\therefore f(z) = \frac{1-i}{1+i} z^3 + \frac{C}{1+i}$

$\therefore f(z) = -i z^3 + c_1$

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* **Ex:** Show that an analytic function with constant modulus is constant.

Soln: Let an analytic function be $f(z) = u + iv$.

$\therefore |f(z)|^2 = u^2 + v^2$

Given, $|f(z)| =$ Constant $= c$ (say) $(c \neq 0)$

i.e, $u^2 + v^2 = c^2$

$\therefore u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0$ ---- (1)

and $u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$ ---- (2)

i.e $-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0$ ---- (2) [By Cauchy-Riemann eqn $u_y = -v_x$, $v_y = u_x$]

Squaring and adding (1) and (2) we have

$(u^2+v^2)\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\} = 0$

i.e, $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0$ $(\because u^2 + v^2 = c^2 \neq 0)$

i.e, $|f'(z)|^2 = 0$ (Since, $f'(z) = u_x + iv_x$)

i.e, $f'(z) = 0$

Hence, $f(z) =$ Constant.

* **Ex:** If $f(z)$ is an analytic function of z, then prove that

i) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

ii) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Re f(z)|^2 = 2|f'(z)|^2$

iii) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

Proof: We have, $z = x + iy$ $\therefore \bar{z} = x - iy$

$\therefore x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$

Now, $f = f(x, y)$

$\therefore \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$

$\therefore \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$

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$$\therefore \frac{\partial f_{\bar{z}}}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \bar{z}}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

$$= \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$

$$= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f.$$

$$\therefore \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}. \quad \text{[Proved]}$$

ii) Now, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |Re f(z)|^2$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}} |Re f(z)|^2$$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}} \left|\frac{1}{2}\{f(z) + \overline{f(z)}\}\right|^2$$

$$= \frac{\partial^2}{\partial z \partial \bar{z}} \{f(z) + \overline{f(z)}\}^2$$

$$= \frac{\partial}{\partial z}\{2\{f(z) + \overline{f(z)}\}\overline{f'(z)}\}$$

$$= 2 f'(z)\overline{f'(z)}$$

$$= 2 |f'(z)|^2 \quad [\because |f(z)|^2 = f(z)\overline{f(z)}]$$

[Proved]

Since $f(z) = u + iv$

$\overline{f(z)} = u - iv$

$\therefore u = \frac{1}{2}\{f(z) + \overline{f(z)}\}$

$= Re f(z)$

iii) Now, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4\frac{\partial^2}{\partial z \partial \bar{z}} |f(z)|^2$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}}\{f(z)\overline{f(z)}\}$$

$$= 4\frac{\partial}{\partial z} f(z) f'(\bar{z})$$

$$= 4 f'(z) f'(\bar{z})$$

$$= 4 |f'(z)|^2$$

[Proved].

* Ex: Show that a harmonic function satisfies the formal differential equation $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$.

Sol^n: We know $z = x + iy$ $\therefore \bar{z} = x - iy$

$\therefore x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$

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Now, $\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z}$

$= \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y}$

$= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$

Similarly, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$\therefore \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

$\Rightarrow \frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

$= \frac{1}{4} \times 0$ [$\because u$ is a harmonic function]

$= 0$

* Ex: If $f(z) = u + iv$ is a function of z in any domain prove that

i) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2 |f(z)|^{p-2} |f'(z)|^2$

ii) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|u|^p = p(p-1)|u|^{p-2}|f'(z)|^2$

Proof: We know that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

i) Now, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p$

$= 4\frac{\partial^2}{\partial z \partial \bar{z}}|f(z)|^p$

$= 4\frac{\partial^2}{\partial z \partial \bar{z}}\{f(z)\overline{f(z)}\}^{p/2}$

$= 4\frac{\partial}{\partial z}\left[\{f(z)\}^{p/2} \cdot p/2\{f(\bar{z})\}^{p/2-1} f'(\bar{z})\right]$

$= 4\left[p/2\{f(z)\}^{p/2-1} f'(z) \cdot p/2\{f(\bar{z})\}^{p/2-1} f'(\bar{z})\right]$

$= p^2\{f(z)f(\bar{z})\}^{p/2-1}\{f'(z)f'(\bar{z})\}$

$= p^2\{|f(z)|^2\}^{p/2-1}|f'(z)|^2$

$= p^2|f(z)|^{p-2}|f'(z)|^2$ [Proved]

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ii) Since, $u+iv = f(z)$ and $u-iv = f(\bar{z})$

$\therefore u = \frac{1}{2}\{f(z)+f(\bar{z})\}$

Now, $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|u|^p$

$= 4\frac{\partial^2}{\partial z \partial \bar{z}}\left|\frac{1}{2}\{f(z)+f(\bar{z})\}\right|^p$

$= \frac{4}{2^p}\frac{\partial^2}{\partial z \partial \bar{z}}\left|\{f(z)+f(\bar{z})\}^2\right|^{p/2}$

$= \frac{4}{2^p}\frac{\partial^2}{\partial z \partial \bar{z}}\cdot\left[\{f(z)+f(\bar{z})\}\cdot\{f(\bar{z})+f(z)\}\right]^{p/2}$

$= \frac{4}{2^p}\frac{\partial^2}{\partial z \partial \bar{z}}\{f(z)+f(\bar{z})\}^p$

$= \frac{4}{2^p}\frac{\partial}{\partial z}\, p\{f(z)+f(\bar{z})\}^{p-1} f'(\bar{z})$

$= \frac{4}{2^p}\, p(p-1)\{f(z)+f(\bar{z})\}^{p-2} f'(z) f'(\bar{z})$

$= \frac{4}{2^p}\cdot p(p-1)\left[\{f(z)+f(\bar{z})\}^2\right]^{\frac{p-2}{2}}|f'(z)|^2$

$= \frac{4}{2^p}\, p(p-1)\left[\{f(z)+f(\bar{z})\}\{f(\bar{z})+f(z)\}\right]^{\frac{p-2}{2}}|f'(z)|^2$

$= \frac{4}{2^{p-2}}p(p-1)\left[|f(z)+f(\bar{z})|^2\right]^{\frac{p-2}{2}}|f'(z)|^2$

$= p(p-1)\left[\left|\frac{1}{2}\{f(z)+f(\bar{z})\}\right|^2\right]^{\frac{p-2}{2}}|f'(z)|^2$

$= p(p-1)\left[|u|^2\right]^{\frac{p-2}{2}}|f'(z)|^2$

$= p(p-1)|u|^{p-2}|f'(z)|^2$. [Proved]

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