Theorem-18:- Let f be bounded and Riemann integrable in $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$, $a \le x \le b$, Prove that F is Continuous in $[a,b]$.

Furthermore, if f be continuous at a point x_0 of $[a,b]$, then show that, F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Proof:- V.H → 199, 202

We have, $F(x) = \int_a^x f(t)\,dt \longrightarrow (1)$, $a \le x \le b$.

Then if $x+h$ $(h \neq 0)$ be a point of $[a,b]$,

$$\therefore F(x+h) = \int_a^{x+h} f(t)\,dt$$

$$\text{Now, } F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt$$

Now, Since f is integrable on $[x, x+h]$. Then there exist a number μ satisfying $m \le \mu \le M$ where m and M are the greatest lower bound and least upper bound of $f(t)$ in $[x, x+h]$ such that

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt = \mu(x+h-x) \quad \left[\begin{array}{l}\text{By 1st M.V.T} \\ \int_a^b f(x)\,dx = \mu(b-a)\end{array}\right.$$

$$= \mu h$$

$$\therefore |F(x+h) - F(x)| = |\mu h|$$

$$= |\mu|\,|h|$$

Since, $f(t)$ is bounded on $[a,b]$, we can choose that $|\mu| \le K$, K being the supremum of $|f(t)|$ on $[a,b]$. Hence,

$$|F(x+h) - F(x)| \le K|h| < \epsilon \quad \text{if } |h| < \frac{\epsilon}{K} = \delta$$

$\therefore$ Thus, $F(x)$ is continuous and also uniformly continuous on $[a,b]$.

[Proved]

2nd part:- We have, $F(x+h) - F(x) = \mu h$

$$\Rightarrow \frac{F(x+h)-F(x)}{h} = \mu$$

$$\Rightarrow \frac{F(x+h)-F(x)}{h} = f(\xi)$$ [Since, $f(x)$ is continuous on $[a,b]$ and $\mu = f(\xi)$, $x \le \xi \le x+h$]

$$\Rightarrow \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} f(\xi)$$

$$= \lim_{h\to 0} f(x+\theta h)$$ [where $\xi = x+\theta h$, $\forall x \in [a,b]$ and $0<\theta<1$]

$$\Rightarrow F'(x) = f(x), \quad \forall x \in [a,b]$$

$\therefore$ $F(x)$ is differentiable on $[a,b]$ also

$$F'(x_0) = f(x_0), \quad \forall x_0 \in [a,b] \quad \text{[Proved]}.$$

Theorem-19:-

What is meant by primitive of a function? Prove that every function continuous in $[a,b]$, possesses a primitive.

V.H - '02

Solⁿ:- PRIMITIVE:- Any function $F(x)$ whose derivative is equal to $f(x)$ at every point x in an interval $[a,b]$ is called a primitive of $f(x)$ or an indefinite integral of $f(x)$ in the interval. i.e $F'(x) = f(x)$, $\forall x \in [a,b]$

2nd For example, $\frac{d}{dx}(\sin x) = \cos x$

$\therefore$ $\sin x$ is called the primitive of $\cos x$

2nd part:- Same as the theorem-18.

Theorem-20 Second Mean Value Theorem for Integral:-

i) Bonnet's Form:- Let $f(x)$ be bounded monotone non-increasing and never negative on $[a,b]$, and let

$\phi(x)$ be bounded and integrable on $[a,b]$. Then there exists a value ξ of x on $[a,b]$, such that

$$\int_a^b f(x)\phi(x)\,dx = f(a)\int_a^{\xi}\phi(x)\,dx \quad , \quad a \le \xi \le b$$

ii) **Weierstrass Form:-** Let, $f(x)$ be bounded and monotonic on $[a,b]$ and let $\phi(x)$ be bounded and integrable on $[a,b]$. Then there exists at least one value of x, say ξ on $[a,b]$ such that

$$\int_a^b f(x)\phi(x)\,dx = f(a)\int_a^{\xi}\phi(x)\,dx + f(b)\int_{\xi}^{b}\phi(x)\,dx \; ; a \le \xi \le b$$

* **Ex-1:-** Show that the function defined by $f(x) = \frac{1}{2^n}$ for $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$, $n = 0,1,2,\ldots$ and $f(x) = 0$, $x = 0$ is integrable over the interval $[0,1]$, although it has an infinite number of points of discontinuity. and also show that $\int_0^1 f(x)\,dx = \frac{2}{3}$. V. H→'92, C. H-'94,'96

Soln:- The given function $f(x) = \frac{1}{2^n}$; $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$

$= 0$; $x = 0$

Here,
$f(0) = 0$
$f(x) = 1$ when $\frac{1}{2} < x \le 1$
$f(x) = \frac{1}{2}$ when $\frac{1}{2^2} < x \le \frac{1}{2}$
$f(x) = \frac{1}{2^2}$ when $\frac{1}{2^3} < x \le \frac{1}{2^2}$
$f(x) = \frac{1}{2^3}$ when $\frac{1}{2^4} < x \le \frac{1}{2^3}$
- - - - - - - - -
$f(x) = \frac{1}{2^{n-1}}$ when $\frac{1}{2^n} < x \le \frac{1}{2^{n-1}}$
- - - - - - - - -

The given function $f(x)$ is discontinuous at the points $0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots$

The set of points of discontinuities has only one limit point 0.

Thus, f is continuous on $[0,1]$ except at the set of points $0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots, \frac{1}{2^n}, \cdots$ which has only one limiting point 0 and f is bounded on $[0,1]$

Hence, $f(x)$ is integrable on $[0,1]$.

2nd part:- $\int_{1/2^n}^{1} f(x)dx = \int_{1/2}^{1} f(x)dx + \int_{1/2^2}^{1/2} f(x)dx + \int_{1/2^3}^{1/2^2} f(x)dx + \cdots + \int_{1/2^n}^{1/2^{n-1}} f(x)dx$

$$= \int_{1/2}^{1} dx + \int_{1/2^2}^{1/2} \frac{1}{2}dx + \int_{1/2^3}^{1/2^2} \frac{1}{2^2}dx + \cdots + \int_{1/2^n}^{1/2^{n-1}} \frac{1}{2^{n-1}}dx$$

$$= \left(1-\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}-\frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^2}-\frac{1}{2^3}\right) + \cdots + \frac{1}{2^{n-1}}\left(\frac{1}{2^{n-1}}-\frac{1}{2^n}\right)$$

$$= \frac{1}{2}\left[1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + \cdots + \left(\frac{1}{2^2}\right)^{n-1}\right]$$

$$= \frac{1}{2} \cdot \frac{1-\left(\frac{1}{2^2}\right)^n}{1-\frac{1}{2^2}}$$

$$= \frac{1}{2} \cdot \frac{1-\frac{1}{4^n}}{1-\frac{1}{4}}$$

$$= \frac{1}{2} \cdot \frac{4}{3}\left(1-\frac{1}{4^n}\right)$$

$$= \frac{2}{3}\left(1-\frac{1}{4^n}\right)$$

Now, taking $n \to \infty$ we get $\int_0^1 f(x)dx = \frac{2}{3}$. [Proved]

* **Ex. 2:-** State Bonnet's form of second mean value theorem and use it to show that

$$\left|\int_a^b \frac{\sin x}{x} dx\right| \le \frac{2}{a}, \quad b > a > 0$$

V. H- '92, '01

Solⁿ:- Statement of Bonnet's form of Second MVT:-

Let, $f(x)$ be bounded monotonic non-increasing and never negative on $[a,b]$ and let $\phi(x)$ be bounded and integrable on $[a,b]$. Then there exist a value of ξ of x on $[a,b]$ such that $\int_a^b f(x)\,\phi(x)\,dx = f(a)\int_a^{\xi}\phi(x)\,dx$; $a \le \xi \le b$

2nd part:-

Let, $f(x) = \frac{1}{x}$ and $\phi(x) = \sin x$

Since, $f(x)$ be bounded monotone non-increasing and never negative on $[a,b]$. And $\phi(x)$ be bounded and integrable on $[a,b]$.

Now, by using 2nd M.V.T of Bonnet's form we get

$$\int_a^b f(x)\,\phi(x)\,dx = f(a)\int_a^{\xi}\phi(x)\,dx \quad ; a \le \xi \le b$$

$$\therefore \int_a^b \frac{\sin x}{x}\,dx = \frac{1}{a}\int_a^{\xi}\sin x\,dx$$
$$= \frac{1}{a}\left[-\cos x\right]_a^{\xi}$$
$$= \frac{1}{a}\left[\cos a - \cos\xi\right]$$

2) $$\left|\int_a^b \frac{\sin x}{x}\,dx\right| = \frac{1}{a}\left|\cos a - \cos\xi\right|$$
$$\le \frac{1}{a}\left\{|\cos a| + |\cos\xi|\right\}$$
$$\le \frac{1}{a}(1+1) \quad \left[\because |\cos a| \le 1,\ \because |\cos\xi| \le 1\right]$$
$$= \frac{2}{a}$$

$$\therefore \left|\int_a^b \frac{\sin x}{x}\,dx\right| \le \frac{2}{a} \quad ; \ b > a > 0$$

[Proved]

*** Ex. 9:-** Apply 2nd M.V.T of Weierstrass's form to prove

$$\left|\int_a^b \frac{\sin x}{x}\,dx\right| < \frac{4}{a} \quad ; \ 0 < a < b$$

Solⁿ:- Let, $f(x) = \frac{1}{x}$ be bounded and monotonic on $[a,b]$ and $\phi(x) = \sin x$ be bounded and integrable on $[a,b]$

Therefore, by 2nd M.V.T of Weierstrass's form of integral calculus we get

$$\int_a^b f(x)\,\phi(x)\,dx = f(a)\int_a^{\xi}\phi(x)\,dx + f(b)\int_{\xi}^{b}\phi(x)\,dx \quad , a \le \xi \le b$$

$$\therefore \int_a^b \frac{\sin x}{x}dx = \frac{1}{a}\int_a^{\xi} \sin x\,dx + \frac{1}{b}\int_{\xi}^{b} \sin x\,dx$$

$$= \frac{1}{a}\left[-\cos x\right]_a^{\xi} + \frac{1}{b}\left[-\cos x\right]_{\xi}^{b}$$

$$= \frac{1}{a}\left[\cos a - \cos\xi\right] + \frac{1}{b}\left[\cos\xi - \cos b\right]$$

$$\therefore \left|\int_a^b \frac{\sin x}{x}dx\right| = \left|\frac{1}{a}(\cos a - \cos\xi) + \frac{1}{b}(\cos\xi - \cos b)\right|$$

$$\leq \frac{1}{a}\left|\cos a - \cos\xi\right| + \frac{1}{b}\left|\cos\xi - \cos b\right|$$

$$\leq \frac{1}{a}\left(|\cos a| + |\cos\xi|\right) + \frac{1}{b}\left(|\cos\xi| + |\cos b|\right)$$

$$\leq \frac{1}{a}(1+1) + \frac{1}{b}(1+1) \qquad \left[\because |\cos x| \leq 1\ \forall x \in [a,b]\right]$$

$$= \frac{2}{a} + \frac{2}{b}$$

$$< \frac{2}{a} + \frac{2}{a} \qquad \left[\because a < b \quad \therefore \frac{2}{a} > \frac{2}{b}\right]$$

$$= \frac{4}{a}$$

$$\therefore \left|\int_a^b \frac{\sin x}{x}dx\right| < \frac{4}{a}, \quad 0 < a < b \qquad \text{[Proved].}$$

* Ex-4:- Show that $\frac{\sin x}{x}$ is R-integrable in $[\pi/4, \pi/3]$ hence show that $\frac{\sqrt{3}}{8} \leq \int_{\pi/4}^{\pi/3} \frac{\sin x}{x}dx \leq \frac{\sqrt{2}}{6}$

V.H → '96, '02
C.H → '93

From the same inequality, prove that

$$\int_0^{\pi/2} e^{-R\sin x}dx < \int_0^{\pi/2} e^{-\frac{2R}{\pi}x}dx = \frac{\pi}{2R}\left(1 - e^{-R}\right)$$

Solnː- Let, $f(x) = \frac{\sin x}{x}$

The function $f(x)$ is continuous on $[\pi/4, \pi/3]$. So it is integrable there.

Now, $f'(x) = \frac{x\cos x - \sin x}{x^2} < 0$ in $[\pi/4, \pi/3]$

So, $f(x)$ is monotonic non-increasing function in $[\pi/4, \pi/3]$

Again $f(x)$ is continuous on $[\pi/4, \pi/3]$.

So, $m =$ minimum value of $f(x) = f(\pi/3)$

$$= \frac{\sin \pi/3}{\pi/3} = \frac{3\sqrt{3}}{2\pi}$$

and $M =$ maximum value of $f(x) = f(\pi/4) = \frac{\sin \pi/4}{\pi/4}$

$$= \frac{4}{\sqrt{2}\pi} = \frac{2\sqrt{2}}{\pi}$$

We know, $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

$$\Rightarrow \frac{3\sqrt{3}}{2\pi}(\pi/3 - \pi/4) \leq \int_{\pi/4}^{\pi/3} \frac{\sin x}{x}dx \leq \frac{2\sqrt{2}}{\pi}(\pi/3 - \pi/4)$$

$$\Rightarrow \frac{3\sqrt{3}}{2\pi} \cdot \frac{\pi}{12} \leq \int_{\pi/4}^{\pi/3} \frac{\sin x}{x}dx \leq \frac{2\sqrt{2}}{\pi} \cdot \frac{\pi}{12}$$

$$\Rightarrow \frac{\sqrt{3}}{8} \leq \int_{\pi/4}^{\pi/3} \frac{\sin x}{x}dx \leq \frac{\sqrt{2}}{6}$$ [Proved]

<u>2nd part:-</u>

We consider a function, $f(x) = \frac{\sin x}{x}$

$$\therefore f'(x) = \frac{x\cos x - \sin x}{x^2} < 0 \text{ in } [0, \pi/2]$$

Since, the function 'f' is monotone non-increasing on $[0, \pi/2]$

Again, $0 < x < \pi/2$

i.e, $f(0) > f(x) > f(\pi/2)$ [As the function is monotone non-increasing on $[0, \pi/2]$]

i.e, $f(x) > f(\pi/2)$

i.e, $\frac{\sin x}{x} > \frac{2}{\pi}\sin \pi/2$

i.e, $\sin x > \frac{2x}{\pi}$

i.e, $-R\sin x < -\frac{2R}{\pi}x$

i.e, $e^{-R\sin x} < e^{-\frac{2R}{\pi}x}$

Since, each of the term is continuous on $[0, \pi/2]$. So, they are integrable on $[0, \pi/2]$

$$\therefore \int_0^{\pi/2} e^{-R\sin x}dx < \int_0^{\pi/2} e^{-\frac{2R}{\pi}x}dx$$ [Proved]

Now, $\int_0^{\pi/2} e^{-\frac{2R}{\pi}x}dx = -\frac{\pi}{2R}\left[e^{-\frac{2R}{\pi}x}\right]_0^{\pi/2}$

$$= -\frac{\pi}{2R}(e^{-R} - 1) = \frac{\pi}{2R}(1 - e^{-R})$$ [Proved]

* **Ex. 5:-** Show that $4 \le \int_1^3 \sqrt{3+x^3}\,dx \le 2\sqrt{30}$

Solⁿ:-

Let, $f(x) = \sqrt{3+x^3}$

$\therefore f'(x) = \dfrac{1}{2\sqrt{3+x^3}} 3x^2 > 0$ in $[1,3]$

$\therefore f(x)$ is monotone increasing in $[1,3]$

Since, $f(x)$ is continuous on $[1,3]$ and monotonic increasing in $[1,3]$.

Now, $M =$ maximum value of $f(x) = f(3) = \sqrt{3+27} = \sqrt{30}$

$m =$ minimum value of $f(x) = f(1) = \sqrt{3+1} = 2$

We have, $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

$\Rightarrow 2(3-1) \le \int_1^3 \sqrt{3+x^3}\,dx \le \sqrt{30}(3-1)$

$\Rightarrow 4 \le \int_1^3 \sqrt{3+x^3}\,dx \le 2\sqrt{30}$ [Proved].

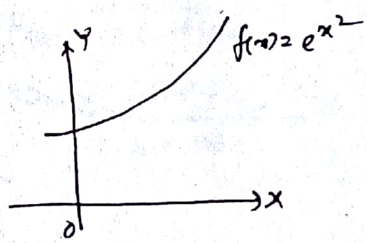
* **Ex. 6:-** Show that $1 \le \int_0^1 e^{x^2}\,dx < e$

Solⁿ:- We have, $0 < x < 1$

$\therefore 0 < x^2 < 1$

$\Rightarrow e^0 < e^{x^2} < e$

$\Rightarrow 1 < e^{x^2} < e$



Since, each of term is continuous in $[0,1]$. So, they are integrable in $[0,1]$

$\therefore \int_0^1 dx < \int_0^1 e^{x^2}\,dx < \int_0^1 e\,dx$

$\Rightarrow 1 < \int_0^1 e^{x^2}\,dx < e$ [Proved]

* **Ex. 7:-** Prove that $\left| \int_{10}^{19} \frac{\sin x}{1+x^8}\,dx \right| < \frac{1}{10^7}$.

Solⁿ:-

$\left| \int_{10}^{19} \frac{\sin x}{1+x^8}\,dx \right| \le \int_{10}^{19} \left| \frac{\sin x}{1+x^8} \right| dx$ $\left[\because \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx\right.$

$\le \int_{10}^{19} \frac{dx}{1+x^8}$ $[\because |\sin x| \le 1]$

Let, $f(x) = \frac{1}{1+x^8}$ $\therefore f'(x) = \frac{-8x^7}{(1+x^8)^2} < 0$ in $[10, 19]$

~~Now~~, So, it is decreasing function in $[10,19]$. Since, it is continuous in the interval $[10,19]$

Now, $M =$ maximum value of $f(x)$

$$= f(10)$$
$$= \frac{1}{1+10^8}$$

$$\therefore \left| \int_{10}^{19} \frac{\sin x}{1+x^8} dx \right| \leq \int_{10}^{19} \frac{dx}{1+10^8}$$
$$\leq M(b-a)$$
$$= \frac{1}{1+10^8}(19-10)$$
$$= \frac{9}{1+10^8} < \frac{1}{10^7} \quad [\because 1+10^8 > 9 \cdot 10^7]$$

$$\therefore \left| \int_{10}^{19} \frac{\sin x}{1+x^8} dx \right| < \frac{1}{10^7} \quad \text{[Proved]}$$

* **Ex. 8:-** If $0 \leq x \leq 1$, show that $\frac{x^2}{\sqrt{2}} \leq \frac{x^2}{\sqrt{1+x}} \leq x^2$ and hence deduce that $\frac{1}{3\sqrt{2}} \leq \int_0^1 \frac{x^2 dx}{\sqrt{1+x}} \leq \frac{1}{3}$. C.H → 1994

Solⁿ:- We have, $0 \leq x \leq 1$

$$\Rightarrow 1 \leq 1+x \leq 2$$
$$\Rightarrow 1 \leq \sqrt{1+x} \leq \sqrt{2}$$
$$\Rightarrow 1 \geq \frac{1}{\sqrt{1+x}} \geq \frac{1}{\sqrt{2}}$$
$$\Rightarrow x^2 \geq \frac{x^2}{\sqrt{1+x}} \geq \frac{x^2}{\sqrt{2}}$$

$$\text{i.e. } \frac{x^2}{\sqrt{2}} \leq \frac{x^2}{\sqrt{1+x}} \leq x^2 \quad \text{[Proved]}$$

Since, each of the term is continuous on $[0,1]$ and then they are integrable on $[0,1]$.

$$\therefore \int_0^1 \frac{x^2}{\sqrt{2}} dx \leq \int_0^1 \frac{x^2}{\sqrt{1+x}} dx \leq \int_0^1 x^2 dx$$

$$\Rightarrow \left[\frac{x^3}{3\sqrt{2}} \right]_0^1 \le \int_0^1 \frac{x^2}{\sqrt{1+x}}\,dx \le \left[\frac{x^3}{3} \right]_0^1$$

$$\Rightarrow \frac{1}{3\sqrt{2}} \le \int_0^1 \frac{x^2}{\sqrt{1+x}}\,dx \le \frac{1}{3}$$ [Proved].

*** Ex-9:-** Prove that, $\frac{2\pi^2}{9} \le \int_{\pi/6}^{\pi/2} \frac{2x}{\sin x}\,dx \le \frac{4\pi^2}{9}$ C. H.-'98, V. H. - 2000

Solⁿ:- We have, $\pi/6 \le x \le \pi/2$

$$\Rightarrow \sin \pi/6 \le \sin x \le \sin \pi/2$$

$$\Rightarrow \frac{1}{2} \le \sin x \le 1$$

$$\Rightarrow 2 \ge \frac{1}{\sin x} \ge 1$$

$$\Rightarrow 4x \ge \frac{2x}{\sin x} \ge 2x$$

Since, each term is continuous in $[\pi/6, \pi/2]$ and so they are integrable in $[\pi/6, \pi/2]$

$$\therefore 4\int_{\pi/6}^{\pi/2} x\,dx \ge \int_{\pi/6}^{\pi/2} \frac{2x}{\sin x}\,dx \ge 2\int_{\pi/6}^{\pi/2} x\,dx$$

$$\Rightarrow 4\left[\frac{x^2}{2}\right]_{\pi/6}^{\pi/2} \ge \int_{\pi/6}^{\pi/2} \frac{2x}{\sin x}\,dx \ge 2\left[\frac{x^2}{2}\right]_{\pi/6}^{\pi/2}$$

$$\Rightarrow 2\left(\frac{\pi^2}{4} - \frac{\pi^2}{36}\right) \ge \int_{\pi/6}^{\pi/2} \frac{2x}{\sin x}\,dx \ge 2\left(\frac{\pi^2}{4} - \frac{\pi^2}{36}\right)$$

$$\Rightarrow \frac{4\pi^2}{9} \ge \int_{\pi/6}^{\pi/2} \frac{2x}{\sin x}\,dx \ge \frac{2\pi^2}{9}$$

$$\therefore \frac{2\pi^2}{9} \le \int_{\pi/6}^{\pi/2} \frac{2x}{\sin x}\,dx \le \frac{4\pi^2}{9}$$ [Proved].

*** Ex-10:-** Show that $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \pi/6$. V. H - '95

Solⁿ:- Since, $0 < x < 1$

$$\Rightarrow 0 < x^3 < x^2$$

$$\Rightarrow -x^2 < -x^2 + x^3 < 0$$

$$\Rightarrow 4 - x^2 < 4 - x^2 + x^3 < 4$$

$\Rightarrow \sqrt{4-x^2} < \sqrt{4-x^2+x^3} < 2$

$\Rightarrow \frac{1}{2} < \frac{1}{\sqrt{4-x^2+x^3}} < \frac{1}{\sqrt{4-x^2}}$

Since each of the term $\frac{1}{2}$, $\frac{1}{\sqrt{4-x^2+x^3}}$, $\frac{1}{\sqrt{4-x^2}}$ are Continuous in $[0,1]$. So they are integrable in $[0,1]$

$\therefore \int_0^1 \frac{1}{2} dx < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \int_0^1 \frac{dx}{\sqrt{4-x^2}}$

$\Rightarrow \frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \left[\sin^{-1}\frac{x}{2}\right]_0^1$

$\Rightarrow \frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \sin^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow \frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \pi/6$ [Proved]

* **Ex-11:-** If $n>1$, prove that $0.5 < \int_0^{1/2} \frac{dx}{\sqrt{1-x^{2n}}} < 0.524$

C.H- '88

Proof:- Since, $x > x^n$ $(n>1)$

$\Rightarrow x^2 > x^{2n}$

$\Rightarrow -x^2 < -x^{2n}$

$\Rightarrow 1-x^2 < 1-x^{2n}$

$\Rightarrow \sqrt{1-x^2} < \sqrt{1-x^{2n}}$

$\Rightarrow \frac{1}{\sqrt{1-x^2}} > \frac{1}{\sqrt{1-x^{2n}}} \longrightarrow (1)$

And, $x^{2n} > 0 \Rightarrow -x^{2n} < 0$

$\Rightarrow 1-x^{2n} < 1$

$\Rightarrow \sqrt{1-x^{2n}} < 1$

$\Rightarrow \frac{1}{\sqrt{1-x^{2n}}} > 1 \longrightarrow (2)$

From (1) and (2) we get

$1 < \frac{1}{\sqrt{1-x^{2n}}} < \frac{1}{\sqrt{1-x^2}} \longrightarrow (3)$

Since, each term of (3) are continuous in $[0, \frac{1}{2}]$ and so they are integrable in $[0, \frac{1}{2}]$.

$$\therefore \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} > \int_0^{1/2} \frac{dx}{\sqrt{1-x^{2n}}} > \int_0^{1/2} dx$$

$$\Rightarrow \sin^{-1}x \Big]_0^{1/2} > \int_0^{1/2} \frac{dx}{\sqrt{1-x^{2n}}} > \frac{1}{2}$$

$$\Rightarrow 0.5 < \int_0^{1/2} \frac{dx}{\sqrt{1-x^{2n}}} < \pi/6$$

$$\Rightarrow 0.5 < \int_0^{1/2} \frac{dx}{\sqrt{1-x^{2n}}} < 0.524 \quad \text{[Proved]}.$$

[Since, $\frac{\pi}{6} = \frac{3.1416}{6} = 0.524$ (approx)]

* **Ex-12:-** Prove that, $0.573 < \int_1^2 \frac{dx}{\sqrt{4-3x+x^3}} < 0.595$

Soln:- Put, $x = 1+u$

$$\therefore 4-3x+x^3 = 4-3(1+u)+(1+u)^3$$
$$= 4-3-3u+1+3u+3u^2+u^3$$
$$= u^3+3u^2+2$$

Now, $1 < x < 2$

$$\Rightarrow 1 < 1+u < 2$$
$$\Rightarrow 0 < u < 1$$
$$\Rightarrow 0 < u^3 < u^2$$
$$\Rightarrow 3u^2 < u^3+3u^2 < 4u^2$$
$$\Rightarrow 2+3u^2 < u^3+3u^2+2 < 2+4u^2$$
$$\Rightarrow \sqrt{2+3u^2} < \sqrt{u^3+3u^2+2} < \sqrt{2+4u^2}$$
$$\Rightarrow \frac{1}{\sqrt{2+4u^2}} < \frac{1}{\sqrt{u^3+3u^2+2}} < \frac{1}{\sqrt{2+3u^2}}$$

Since each term is continuous on $[0,1]$ and so they are integrable on $[0,1]$.

$$\therefore \int_0^1 \frac{du}{\sqrt{2+4u^2}} < \int_0^1 \frac{du}{\sqrt{u^3+3u^2+2}} < \int_0^1 \frac{du}{\sqrt{2+3u^2}}$$

$$\Rightarrow \frac{1}{2}\left[\log\left|u+\sqrt{u^2+\frac{1}{2}}\right|\right]_0^1 < \int_0^1 \frac{du}{\sqrt{u^3+3u^2+2}} < \frac{1}{\sqrt{3}}\left[\log\left|u+\sqrt{u^2+\frac{2}{3}}\right|\right]_0^1$$

$$\Rightarrow \frac{1}{2}\log\left(\frac{1+\sqrt{\frac{3}{2}}}{\sqrt{\frac{1}{2}}}\right) < \int_0^1 \frac{du}{\sqrt{u^3+3u^2+2}} < \frac{1}{\sqrt{3}}\log\left|\frac{1+\sqrt{\frac{5}{3}}}{\sqrt{\frac{2}{3}}}\right|$$

$$\Rightarrow 0.573 < \int_1^2 \frac{dx}{\sqrt{4-3x+x^3}} < 0.595 \quad [\text{Proved}].$$

* **Ex-13:-** Let, α and ϕ are positive acute angles, then show that

$$\phi < \int_0^\phi \frac{dx}{\sqrt{1-\sin^2\alpha\sin^2 x}} < \frac{\phi}{\sqrt{1-\sin^2\alpha\sin^2\phi}}$$

If $\alpha=\phi=\pi/6$, then prove that, the integral lies between 0.523 and 0.541. **C.H-2000**

Soln:- Since, α and ϕ are acute angles, then

$$0 < x < \phi$$

$$\Rightarrow 0 < \sin^2 x < \sin^2\phi$$

$$\Rightarrow 0 < \sin^2\alpha\sin^2 x < \sin^2\alpha\sin^2\phi$$

$$\Rightarrow 1 > 1-\sin^2\alpha\sin^2 x > 1-\sin^2\alpha\sin^2\phi$$

$$\Rightarrow 1 < \frac{1}{\sqrt{1-\sin^2\alpha\sin^2 x}} < \frac{1}{\sqrt{1-\sin^2\alpha\sin^2\phi}}$$

Since, each term is continuous on $[0,\phi]$, then they are integrable on $[0,\phi]$.

$$\therefore \int_0^\phi dx < \int_0^\phi \frac{dx}{\sqrt{1-\sin^2\alpha\sin^2 x}} < \int_0^\phi \frac{dx}{\sqrt{1-\sin^2\alpha\sin^2\phi}}$$

$$\Rightarrow \phi < \int_0^\phi \frac{dx}{\sqrt{1-\sin^2\alpha\sin^2 x}} < \frac{\phi}{\sqrt{1-\sin^2\alpha\sin^2\phi}} \quad [\text{Proved}]$$

2nd part:- If, $\phi=\alpha=\pi/6$.

$$\therefore \pi/6 < \int_0^\phi \frac{dx}{\sqrt{1-\sin^2\alpha\sin^2 x}} < \frac{\pi/6}{\sqrt{1-\sin^2\pi/6\,\sin^2\pi/6}}$$

$$\Rightarrow 0.523 < \int_0^\phi \frac{dx}{\sqrt{1-\sin^2\alpha\sin^2 x}} < \frac{\pi/6}{\sqrt{1-\frac{1}{16}}}$$

2) $0.523 < \int_0^{\varphi} \frac{dx}{\sqrt{1-\sin^2\alpha \sin^2\varphi}} < 0.541$ [Proved].

* **Ex.-14:-** Prove that $\int_0^1 \frac{x^3 \cos 5x}{2+x^2}\,dx$ lies between $-\frac{1}{2}$ and $\frac{1}{2}$. C.H- 192, 197

Solⁿ:- Now, $\left|\int_0^1 \frac{x^3\cos 5x}{2+x^2}\,dx\right| \le \int_0^1 \left|\frac{x^3 \cos 5x}{2+x^2}\right| dx$ $\left[\because \left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx\right]$

$\le \int_0^1 \left|\frac{x^3}{2+x^2}\right| dx$ $[\because |\cos 5x| \le 1 \ \forall x \in [0,1]]$

$\le \int_0^1 \left|\frac{x^3}{x^2}\right| dx$ $[\because \frac{x^3}{x^2} > \frac{x^3}{2+x^2}$

$= \int_0^1 |x| dx$

$= \frac{1}{2}$

$\therefore -\frac{1}{2} < \int_0^1 \frac{x^3 \cos 5x}{2+x^2}\,dx < \frac{1}{2}$

$\therefore \int_0^1 \frac{x^3\cos 5x}{2+x^2}\,dx$ lies between $-\frac{1}{2}$ and $\frac{1}{2}$.

* **Ex- 15:-** Show that, $\lim_{x\to 3} \frac{1}{x-3}\int_3^x e^{\sqrt{1+t^2}}\,dt = e^{\sqrt{10}}$

Solⁿ:-

Let $\varphi'(t) = e^{\sqrt{1+t^2}}$

R.H.S, $\lim_{x\to 3} \frac{1}{x-3}\int_3^x e^{\sqrt{1+t^2}}\,dt$

$= \lim_{x\to 3} \frac{\int_3^x \varphi'(t)\,dt}{x-3}$

$= \lim_{x\to 3} \frac{\varphi(x) - \varphi(3)}{x-3}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 3} \frac{\varphi'(x)}{1}$

$= \varphi'(3)$

$= e^{\sqrt{1+9}}$

$= e^{\sqrt{10}}$

= R.H.S.

Ex. 16:- Show that $\lim_{x\to 0} \frac{x}{1-e^{x^2}} \int_0^x e^{t^2} dt = -1$ C.H-'91,'00

Soln:- Let, $\phi'(t) = e^{t^2}$

Now,
$$L.H.S = \lim_{x\to 0} \frac{x}{1-e^{x^2}} \int_0^x e^{t^2} dt$$
$$= \lim_{x\to 0} \frac{x}{1-e^{x^2}} \int_0^x \phi'(t) dt$$
$$= \lim_{x\to 0} \frac{x\,[\phi(t)]_0^x}{1-e^{x^2}}$$
$$= \lim_{x\to 0} \frac{x\,[\phi(x)-\phi(0)]}{1-e^{x^2}} \quad (0/0 \text{ form})$$
$$= \lim_{x\to 0} \frac{\phi(x)+x\phi'(x)-\phi(0)}{-2x e^{x^2}} \quad (0/0 \text{ form})$$
$$= \lim_{x\to 0} \frac{\phi'(x)+\phi'(x)+x\phi''(x)}{-2e^{x^2}-4x^2e^{x^2}}$$
$$= \lim_{x\to 0} \frac{2\phi'(x)+x\phi''(x)}{-2e^{x^2}-4x^2e^{x^2}}$$
$$= -1$$
$$= R.H.S \quad [\text{Proved}].$$

* **Ex. 17:-** Use first Mean value theorem to prove that
$$\pi/6 \le \int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{\pi}{6}\cdot\frac{1}{\sqrt{1-\frac{k^2}{4}}}, \quad k^2<1$$

Soln:- Let, $f(x) = \frac{1}{\sqrt{1-k^2x^2}}$, $x\in[0,\frac12]$; $g(x) = \frac{1}{\sqrt{1-x^2}}$; $x\in[0,\frac12]$.

Then, f and g are integrable on $[0,\frac12]$ and $g(x)>0$ for all $x\in[0,\frac12]$.

Since, f is continuous on $[0,\frac12]$, by the first Mean value theorem there exist a point ξ in $[0,\frac12]$ such that
$$\int_0^{1/2} f(x)g(x)dx = f(\xi)\int_0^{1/2} g(x)dx$$
$$\text{or, } \int_0^{1/2} \frac{dx}{\sqrt{(1-k^2x^2)(1-x^2)}} = \frac{1}{\sqrt{1-k^2\xi^2}} \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{\pi}{6}\,\frac{1}{\sqrt{1-k^2\xi^2}}$$

Since, $0 \le \xi \le \frac{1}{2} \Rightarrow 0 \le \xi^2 \le \frac{1}{4}$

$\Rightarrow 0 \le k^2\xi^2 \le \frac{k^2}{4}$

$\Rightarrow 1 \ge 1 - k^2\xi^2 \ge 1 - k^2/4$

$\therefore 1 \le \frac{1}{\sqrt{1-k^2\xi^2}} \le \frac{1}{\sqrt{1-\frac{k^2}{4}}}$

Therefore, $\frac{1}{\sqrt{1-x^2}} \le \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{1}{\sqrt{1-k^2/4}} \frac{1}{\sqrt{1-x^2}}$

Since, each of these term are Continuous on $[0, 1/2]$. and so, they are integrable on $[0, 1/2]$

Therefore, $\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} \le \int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{1}{\sqrt{1-k^2/4}} \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$

$\Rightarrow \left.\sin^{-1}x\right]_0^{1/2} \le \int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \left[\sin^{-1}x\right]_0^{1/2} \frac{1}{\sqrt{1-k^2/4}}$

$\Rightarrow \pi/6 \le \int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{\pi}{6} \frac{1}{\sqrt{1-k^2/4}}$ [Proved].

* **Ex-18:-** Prove that, $\frac{1}{4} < \int_0^{1/4} \frac{dx}{\sqrt{1-x^{2n}}} < \frac{1}{\sqrt{15}}$, V. H-2003

Solⁿ:- Since, as $0 < x < \frac{1}{4}$,

$x^n < x$

$\Rightarrow x^{2n} < x^2$

$\Rightarrow 1 - x^{2n} > 1 - x^2$

$\Rightarrow \frac{1}{\sqrt{1-x^{2n}}} < \frac{1}{\sqrt{1-x^2}} \longrightarrow (1)$

Again $x < \frac{1}{4}$

$\therefore x^2 < \frac{1}{16}$

$\Rightarrow 1 - x^2 > 1 - \frac{1}{16}$

$= \frac{15}{16}$

$\therefore \frac{1}{\sqrt{1-x^2}} < \frac{4}{\sqrt{15}} \longrightarrow (2)$

Also, $\quad x^{2n} > 0$

$\Rightarrow 1 - x^{2n} < 1$

$\Rightarrow \dfrac{1}{\sqrt{1-x^{2n}}} > 1 \longrightarrow (3)$

From (1), (2), (3) we get

$$1 < \frac{1}{\sqrt{1-x^{2n}}} < \frac{1}{\sqrt{1-x^2}} < \frac{4}{\sqrt{15}}$$

i.e, $\quad 1 < \dfrac{1}{\sqrt{1-x^{2n}}} < \dfrac{4}{\sqrt{15}}$

Since, each of the term is continuous in $[0, 1/4]$, then they are integrable in $[0, 1/4]$.

$$\therefore \int_0^{1/4} dx < \int_0^{1/4} \frac{dx}{\sqrt{1-x^{2n}}} < \frac{4}{\sqrt{15}} \int_0^{1/4} dx$$

$$\Rightarrow \frac{1}{4} < \int_0^{1/4} \frac{dx}{\sqrt{1-x^{2n}}} < \frac{4}{\sqrt{15}} \cdot \frac{1}{4}$$

$$\therefore \frac{1}{4} < \int_0^{1/4} \frac{dx}{\sqrt{1-x^{2n}}} < \frac{1}{\sqrt{15}} \quad \text{[Proved]}$$

* **Ex-19:-** Prove that, $\frac{1}{2}(1 - \frac{1}{e}) < \int_0^1 e^{-x^2} dx < 1$ and $0 < \int_1^{\infty} e^{-x^2} dx < \frac{1}{2e}$ and deduce that $\frac{1}{2}(1-\frac{1}{e}) < \int_0^{\infty} e^{-x^2} dx < 1 + \frac{1}{2e}$.

Solⁿ:- In $[0, 1]$, $x < 1$

i.e $\quad x e^{-x^2} < e^{-x^2}$

Since, each term is continuous in $[0,1]$. So they are integrable in $[0,1]$

We have, $\quad \int_0^1 x e^{-x^2} dx < \int_0^1 e^{-x^2} dx$

$$\Rightarrow \frac{1}{2}\left[-e^{-x^2}\right]_0^1 < \int_0^1 e^{-x^2} dx$$

$$\Rightarrow \frac{1}{2}\left[1 - \frac{1}{e}\right] < \int_0^1 e^{-x^2} dx \longrightarrow (1)$$

Again, e^{-x^2} is a decreasing function and for $x=0$

$e^{-x^2} = 1$

$\therefore\ e^{-x^2} < 1$ in $[0,1]$

Since, each term is continuous in $[0,1]$. So they are integrable in $[0,1]$.

$$\therefore \int_0^1 e^{-x^2}dx < \int_0^1 dx$$

$$\therefore \int_0^1 e^{-x^2}dx < 1 \longrightarrow (2)$$

$\therefore$ From (1) and (2) we get

$$\therefore \frac{1}{2}\left(1-\frac{1}{e}\right) < \int_0^1 e^{-x^2}dx < 1 \quad \text{[Proved]} \longrightarrow (3)$$

Again, in $[1,\infty]$, $x>1$

$$\therefore x e^{-x^2} > e^{-x^2}$$

Since each terms is continuous in $[1,\infty)$. So they are integrable in $[1,\infty)$

$$\therefore \int_1^\infty x e^{-x^2}dx > \int_1^\infty e^{-x^2}dx$$

$$\Rightarrow \int_1^\infty e^{-x^2}dx < \frac{1}{2}\left[-e^{-x^2}\right]_1^\infty$$

$$= \frac{1}{2e} \longrightarrow (4)$$

Again, e^{-x^2} being greater than zero i.e $e^{-x^2}>0$

and e^{-x^2} is continuous on $[1,\infty)$. So it is integrable in $[1,\infty)$

$$\therefore \int_1^\infty e^{-x^2}dx > 0 \longrightarrow (5)$$

From (4) & (5) we get, $0 < \int_1^\infty e^{-x^2}dx < \frac{1}{2e} \longrightarrow (6)$ [Proved]

Adding (3) and (6) we get

$$\frac{1}{2}\left(1-\frac{1}{e}\right) < \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx < 1+\frac{1}{2e}$$

i.e $\frac{1}{2}\left(1-\frac{1}{e}\right) < \int_0^\infty e^{-x^2}dx < 1+\frac{1}{2e}$ [Proved].

*** Ex-20 :-** Show that $\int_a^b \frac{|x|}{x}\,dx = |b| - |a| \quad (a<b)$

Solⁿ :- Let, $f(x) = \frac{|x|}{x}$

$= 1 \;;\; x>0$

$= -1 \;;\; x<0$

$=$ undefined $, x=0$

Case-I :- Let, $a<0,\ b>0$

$\therefore \int_a^b f(x)\,dx = \int_a^0 f(x)\,dx + \int_0^b f(x)\,dx$

$= \int_a^0 -dx + \int_0^b dx$

$= b+a$

$= |b| - |a| \qquad [\because |a| = -a,$ as $a<0$

Case-II :- Let, $a>0,\ b>0$

$\therefore \int_a^b f(x)\,dx = \int_a^b dx = [x]_a^b = b-a$

$= |b| - |a|$

Case-III :- Let, $a<0,\ b<0$

$\therefore \int_a^b f(x)\,dx = \int_a^b -dx = [-x]_a^b$

$= a-b$

$= |b| - |a| \qquad$ [Since, $|a| = -a$, $|b| = -b$ as $a,b<0$

$\therefore \int_a^b \frac{|x|}{x}\,dx = |b| - |a| \quad (a<b).$

***. Ex-21 :-** Show that for the function f defined on $0 \le x \le 1$ as follows

$f(x) = \sqrt{1-x^2}$; x is rational

$= 1-x$; x is irrational.

$I = \frac{1}{2}$ and $J = \frac{\pi}{4}$ and so $f(x)$ is not integrable on $[0,1]$.

Solⁿ :- Since, $f(x) = \begin{cases} \sqrt{1-x^2} & ;\ x \text{ is rational.} \\ 1-x & ;\ x \text{ is irrational.} \end{cases}$

Now, $(\sqrt{1-x^2})^2 - (1-x)^2 = 1-x^2-1+2x-x^2$
$= 2x-2x^2$
$= 2x(1-x) > 0$ for all $0 \le x \le 1$

Let, $P = \{0 = x_0, x_1, x_2, \ldots, x_n = 1\}$ be a partition of $[0,1]$ and δ_r be the length of the r-th interval $[x_{r-1}, x_r]$, then in each of the subinterval δ_r, then the upper bound of $f(x) = \sqrt{1-x^2}$ and the lower bound of $f(x) = 1-x$

$$\therefore J = \int_0^1 f(x)dx$$
$$= \int_0^1 \sqrt{1-x^2}\,dx$$
$$= \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$
$$= \frac{1}{2}\sin^{-1}(1)$$
$$= \frac{\pi}{4}$$

$$\therefore I = \int_0^1 f(x)\,dx = \int_0^1 (1-x)dx = \left[x - \frac{x^2}{2}\right]_0^1$$
$$= 1 - \frac{1}{2} = \frac{1}{2}$$

$\therefore I \neq J$

$\therefore$ Hence, $f(x)$ is not R-integrable in $[0,1]$ [Proved].

*** Ex-22:-** Examine the integrability of $f(x)$ over $[0,2]$

where $f(x) = \begin{cases} x^2 & , x \text{ is rational.} \\ x^3 & , x \text{ is irrational.} \end{cases}$

Solⁿ:- Since, $x^2 - x^3 = x^2(1-x) > 0$ for $0 \le x \le 1$
< 0 for $1 \le x \le 2$

Let us consider a partition $P = \{0 = x_0, x_1, x_2, \ldots, x_n = 2\}$ of $[0,2]$ and δ_r denote the length of the r-th interval in $[x_{r-1}, x_r]$

In each of the subinterval the upper bound of $f(x)$ is x^2 in $0 \le x \le 1$ and x^3 in $1 \le x \le 2$.

$$J = \int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx$$
$$= \int_0^1 x^2dx + \int_1^2 x^3dx$$

$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^4}{4}\right]_1^2$

$= \frac{1}{3} + \frac{16}{4} - \frac{1}{4}$

$= \frac{4+48-3}{12} = \frac{49}{12}$

Again each of the subinterval lower bound of $0 \le x \le 1$ is x^3 and $1 \le x \le 2$ is x^2

$\therefore I = \int_0^2 f(x)dx = \int_0^1 x^3 dx + \int_1^2 x^2 dx$

$= \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^3}{3}\right]_1^2$

$= \frac{1}{4} + \frac{8}{3} - \frac{1}{3}$

$= \frac{3+32-4}{12}$

$= \frac{31}{12}$

$\therefore I \neq J$

$\therefore$ Hence, $f(x)$ is not R- integrable in $[0,2]$.

*** Ex-23 :-** Discuss the applicability of the Mean value theorems on the integrals $\int_{-\pi/2}^{\pi/2} x^2 \cos x\, dx$ — C.H- '97, '01

Soln:-

i) **1st MVT:-** Let, $f(x) = \cos x$, $\phi(x) = x^2$

Here $f(x)$ be bounded and integrable on $[-\pi/2, \pi/2]$ and $\phi(x)$ is also bounded and integrable on $[-\pi/2, \pi/2]$ and keeps the same sign (positive) in $[-\pi/2, \pi/2]$.

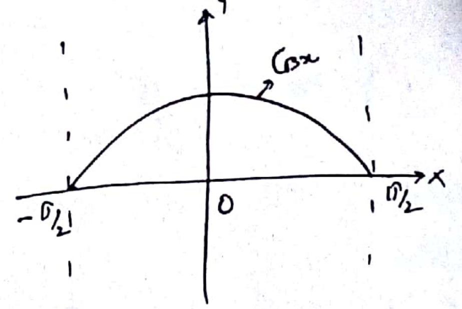
Again, let, $f(x) = x^2$ and $\phi(x) = \cos x$

Here, $f(x)$ be bounded and integrable on $[-\pi/2, \pi/2]$ and $\phi(x)$ be also bounded, integrable and keeps the same sign (positive) in $[-\pi/2, \pi/2]$

Hence, both the cases, the first Mean value theorem is applicable for the integral $\int_{-\pi/2}^{\pi/2} x^2 \cos x\, dx$.

ii) <u>2nd MVT (Bonnet's form)</u> :-

Let, $f(x) = \cos x$, $\phi(x) = x^2$



Here, $f(x)$ be bounded and non-negative in $[-\pi/2, \pi/2]$ and $\phi(x)$ be bounded and integrable on $[-\pi/2, \pi/2]$ but $f(x)$ is not monotonic decreasing.

Since, in $-\pi/2 \leq x \leq 0$, $f(x)$ increases

and in $0 \leq x \leq \pi/2$, $f(x)$ decreases.

Again let, $f(x) = x^2$ and $\phi(x) = \cos x$

Here, $f(x)$ is bounded and non-negative and $\phi(x)$ is bounded and integrable in $[-\pi/2, \pi/2]$ but $f(x)$ is not monotonic decreasing, since

In $-\pi/2 \leq x \leq 0$, x^2 is decreasing

and in $0 \leq x \leq \pi/2$, x^2 is increasing.

Combining two cases, the 2nd MVT in Bonnet's form is not applicable for the integral $\int_{-\pi/2}^{\pi/2} x^2 \cos x \, dx$.

iii) <u>Second MVT (Weierstrass's form)</u>:-

Let $f(x) = \cos x$, $\phi(x) = x^2$

Here, $f(x)$ is bounded but not monotone in $[-\pi/2, \pi/2]$ and $\phi(x)$ be bounded and integrable in $[-\pi/2, \pi/2]$.

Again, $f(x) = x^2$, $\phi(x) = \cos x$

Here, $f(x)$ be bounded but not monotone in $[-\pi/2, \pi/2]$ as x^2 is decreases in $-\pi/2 \leq x \leq 0$ and increases in $0 \leq x \leq \pi/2$.

and $\phi(x)$ be bounded and integrable on $[-\pi/2, \pi/2]$.

Combining both the cases, we can say that the Second MVT is not applicable for the integral $\int_{-\pi/2}^{\pi/2} x^2 \cos x \, dx$.

Ex-24:- Discuss the applicability of the Mean-value theorems on the integral $\int_{-\pi/2}^{\pi/2} x \sin x\, dx$ — C.H-'90, '93

Soln:- i) First MVT:-

Let, $f(x) = \sin x$, $\phi(x) = x$

Here $f(x)$ be bounded and integrable on $[-\pi/2, \pi/2]$ and $\phi(x)$ is also bounded and integrable and does not keep the same sign on $[-\pi/2, \pi/2]$ as $\phi(x) \le 0$ $\forall x \in [-\pi/2, 0]$ and $\phi(x) \geq 0$ $\forall x \in [0, \pi/2]$.

Again let, $f(x) = x$, $\phi(x) = \sin x$

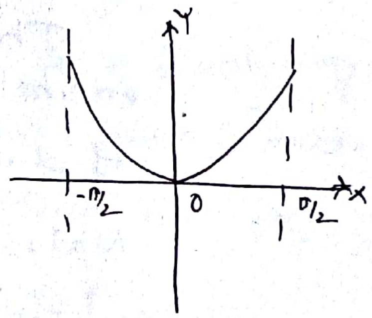
$f(x)$ is also bounded and integrable on $[-\pi/2, \pi/2]$ and $\phi(x)$ is also bounded and integrable but not keep the same sign in $[-\pi/2, \pi/2]$ as $\phi(x) \le 0$ $\forall x \in [-\pi/2, 0]$ and $\phi(x) \geq 0$ $\forall x \in [0, \pi/2]$

Hence, both the cases the First MVT does not applicable on the integral $\int_{-\pi/2}^{\pi/2} x \sin x\, dx$.

ii) Second MVT:- (Bonnet's form)

Let, $f(x) = \sin x$, $\phi(x) = x$

Here, $f(x)$ be bounded, does not always positive and not monotonic decreasing, since in $-\pi/2 \le x \le 0$ where $\sin x$ is decreasing and in $0 \le x \le \pi/2$ where $\sin x$ is increasing



$\phi(x)$ is also bounded and integrable in $[-\pi/2, \pi/2]$

Again, let, $f(x) = x$, $\phi(x) = \sin x$

Since, $f(x)$ is bounded, monotone increasing and does not keep the same sign in $[-\pi/2, \pi/2]$. Also, $\phi(x)$ be bounded and integrable on $[-\pi/2, \pi/2]$.

Combining both the cases, the Second MVT in Bonnet's form is not applicable for the integral $\int_{-\pi/2}^{\pi/2} x \sin x\, dx$.

iii) Second MVT: (Weierstrass's form)

Let, $f(x) = \sin x$ and $\phi(x) = x$

Here, $f(x)$ be bounded but not monotone in $[-\pi/2, \pi/2]$ and $\phi(x)$ be bounded and integrable on $[-\pi/2, \pi/2]$

Again, let, $f(x) = x$ and $\phi(x) = \sin x$

Here, $f(x)$ be bounded but not monotonic in $[-\pi/2, \pi/2]$ and $\phi(x)$ be bounded and integrable on $[-\pi/2, \pi/2]$

Both the cases, the Second MVT of Weierstrass's form is not applicable in the integral $\int_{-\pi/2}^{\pi/2} x \sin x \, dx$.

*** Ex-25:-** Show that $[x]$ is integrable on $[0,3]$ and $\int_0^3 [x] dx = 3$.

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Soln:- On $[0,3]$,

$$[x] = \begin{cases} 0 & ; \quad 0 \le x < 1 \\ 1 & ; \quad 1 \le x < 2 \\ 2 & ; \quad 2 \le x < 3 \\ 3 & ; \quad x = 3 \end{cases}$$

Thus $[x]$ is bounded on $[0,3]$ and is continuous there, except at $x = 1, 2, 3$. Since there are only a finite number of discontinuities in the interval, it is integrable there.

Now, $\int_0^3 [x] dx = \int_0^1 [x] dx + \int_1^2 [x] dx + \int_2^3 [x] dx$

$= \int_0^1 0 \, dx + \int_1^2 dx + \int_2^3 2 \, dx$

$= 0 + (2-1) + 2(3-2)$

$= 3$ [Proved].

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* **Ex-26:-** State fundamental theorem of integral Calculus. Show that evaluation of $\int_0^2 f(x)dx$, where $f(x) = [x]$, $x \in [0,2]$ can not be done by the fundamental theorem of integral Calculus.

V.H - 2000

Solⁿ:- **Fundamental Theorem:-** If $f(x)$ is integrable on the closed interval $[a,b]$ and their exist a function $F(x)$ on $[a,b]$ such that $f(x) = F'(x)$ $\forall x \in [a,b]$ and then $\int_a^b f(x)dx = F(b) - F(a)$.

2nd part:-

Here the function $f(x) = [x] = \begin{cases} 0 , & 0 \le x < 1 \\ 1 , & 1 \le x < 2 \\ 2 , & x = 2 \end{cases}$

The function is discontinuous at $x = 1, 2$, we may consider a function $\phi(x)$ such that $\phi'(x) = f(x)$ as follows

$$\phi(x) = \begin{cases} c , & 0 \le x < 1 \\ x , & 1 \le x < 2 \\ 2x , & x = 2 \end{cases}$$

Where c is an arbitrary constants. As c is arbitrary, then $\phi(x)$ can not be a primitive of $f(x)$. Hence $\int_0^2 f(x)dx$ can not be done by fundamental theorem when $f(x) = [x]$

* **Ex-27:-** State second MVT of integral Calculus in Weierstrass's form and verify it for function $x \sin x$ in $[\pi, 2\pi]$

V.H - 2002

2nd part:- Let, $f(x) = x$ be bounded and monotone on $[\pi, 2\pi]$ and $\phi(x) = \sin x$ be bounded and integrable on $[\pi, 2\pi]$.

Now, using 2nd MVT of ~~for~~ Weierstrass form is applicable and hence,

$$\int_\pi^{2\pi} x \sin x \, dx = f(\pi) \int_\pi^{\xi} \sin x \, dx + f(2\pi) \int_\xi^{2\pi} \sin x \, dx \quad ; \pi \le \xi \le 2\pi$$

$$= \pi \left[-\cos x \right]_\pi^{\xi} + 2\pi \left[-\cos x \right]_\xi^{2\pi}$$

$$= \pi \left[\cos \pi - \cos \xi \right] + 2\pi \left[\cos \xi - \cos 2\pi \right]$$

$$= \pi \left[-1 - \cos \xi \right] + 2\pi \left[\cos \xi - 1 \right]$$

$$= \pi \cos \xi - 3\pi. \longrightarrow (1)$$

Now,
$$\int_\pi^{2\pi} x \sin x \, dx = \left[-x \cos x \right]_\pi^{2\pi} + \int_\pi^{2\pi} \cos x \, dx$$

$$= \left[-2\pi \cos 2\pi + \pi \cos \pi \right] + \left[\sin x \right]_\pi^{2\pi}$$

$$= -2\pi - \pi$$

$$= -3\pi \longrightarrow (2)$$

From (1) and (2) we get

$$-3\pi + \pi \cos \xi = -3\pi$$

$$\therefore \cos \xi = 0 = \cos (2\pi - \pi/2)$$

$$\therefore \xi = \frac{3\pi}{2} \quad \text{where } \pi \le \xi \le 2\pi$$

Hence, Weierstrass form of second MVT for the function is verified in $[\pi, 2\pi]$. ~~(Proved)~~.

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